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Mechanics

1 Introduction and basic concepts

In this section we summarise some basic concepts and methods with which the reader should be familiar. These notes are intended as guidelines and the reader is advised to supplement their studies by reading relevant sections from the many excellent standard texts available. We recommend in particular:

Fundamentals of Physics by Halliday, Resnick & Walker

University Physics by Young & Freedman

There is also much material freely available on the internet. Background material for the physical sciences is available (free of charge) in `pdf` format from the website

http://www.everythingscience.co.za/grade-12

Textbooks for grades 10 and 11, as well as textbooks for mathematics for grades 10–12, are also available.

1.1 Introduction

Physics attempts to describe how and why our universe (including our immediate physical environment) behaves as it does. For example it explains why the sky is blue and why rainbows have colours. It also explains what keeps our moon in its orbit, and accounts for the thunder and lightning that accompany a storm.

The laws of physics are remarkable for their scope, covering the behaviour both of subatomic particles and distant stars far greater than our sun.

It is because physics is so fundamental that it is a required course for students majoring in a wide variety of other subjects.

We hope you will come to see that physics is highly relevant both to you and your environment.

In this course we will study the oldest branch of mechanics called *classical mechanics*. It is used to describe the motion of objects much bigger than atoms moving at speeds much less than the speed of light.

1.2 Units

We shall use the SI (Système International) set of units. This has a number of **base units**, three of which will be important in our study of Mechanics: the metre, the kilogram and the second. **Units must always be given** except in the case of dimensionless quantities.

Physical quantity	Name of S.I. unit	Symbol
Length	metre	m
Mass	kilogram	kg
Time	second	s

Table 1: Several commonly used base units.

Units of quantities such as force and energy can be expressed as combinations of the base units and are referred to as **derived units**. Table 2 lists some of the derived units that will be encountered in this course.

Physical quantity	Name of S.I. unit	Base units	Symbol
Force	newton	$\mathrm{kg\,m\,s^{-2}}$	N
Energy	joule	$\mathrm{kg\,m^2\,s^{-2}}$	J
Pressure	pascal	$\mathrm{N\,m^{-2}}$	Pa

Table 2: Several commonly used derived units.

The value of a quantity is often very large or very small when expressed in base or derived units. It is then convenient to express these quantities in terms of multiples of ten as given by the prefixes summarised in Table 3.

Prefix	Symbol	Factor
Tera	T	10^{12}
Giga	G	10^{9}
Mega	M	10^{6}
Kilo	k	10^{3}
Deci	d	10^{-1}
Centi	c	10^{-2}
Milli	m	10^{-3}
Micro	µ	10^{-6}
Nano	n	10^{-9}
Pico	p	10^{-12}
Femto	f	10^{-15}
Atto	a	10^{-18}

Table 3: Standard prefixes used to denote multiples of ten.

Example 1: Conversion of inches and feet to SI units

There are 12 inches in one foot and one inch is 2.54 cm. Calculate the height of a person who is 6 feet tall in SI units.
Solution:

$$6 \text{ feet} = 6 \times 12 \text{ inches} = 72 \text{ inches} = 72 \times 2.54\,\text{cm} = 183\,\text{cm} = 1.83\,\text{m}.$$

Example 2: Conversion of km/hour to $\text{m}\,\text{s}^{-1}$

Convert $60\,\text{km}\,\text{h}^{-1}$ to $\text{m}\,\text{s}^{-1}$.
Solution:

$$60\,\text{km}\,\text{h}^{-1} = \frac{60\,\text{km}}{1\,\text{h}} = \frac{60 \times 10^3\,\text{m}}{1\,\text{s} \times 60\,\text{s} \times 60\,\text{s}} = 16.7\,\text{m}\,\text{s}^{-1}.$$

Example 3: Conversion of standard prefixes

Express 42 000 km in metres and 3.2×10^{-8} kg in µg.
Solution:

$$42\,000\,\text{km} = 4.2 \times 10^4\,\text{km} = 4.2 \times 10^4 \times 10^3\,\text{m} = 4.2 \times 10^7\,\text{m}.$$
$$3.2 \times 10^{-8}\,\text{kg} = 3.2 \times 10^{-8} \times 10^3\,\text{g} = 3.2 \times 10^{-5} \times 10^6\,\mu\text{g} = 32\,\mu\text{g}.$$

1.3 Scalar and vector quantities

1.3.1 The distinction between scalar and vector quantities

Scalars are quantities that have magnitude only. Scalars are completely specified by the product of a positive or negative number and (usually) a unit. Examples of scalars are mass, temperature, distance and speed. **Vectors** are quantities that have magnitude and direction. The magnitude of a vector is positive. Examples of vectors are force, acceleration, displacement and velocity. Note that a vector (e.g. velocity) changes if **either** its magnitude **or** its direction changes.

There are various ways of indicating that a given symbol is to represent a vector quantity. A force F could be written:

$$\vec{F} \text{ or } \underset{\sim}{F} \text{ or } \mathbf{F} \text{ (i.e. heavy type or boldface).}$$

In these notes vector quantities will be indicated using boldface symbols.

1.3.2 Addition of scalars

Scalars simply add arithmetically. For example, adding several masses:

$$5\,\text{kg} + 10\,\text{kg} + 2\,\text{kg} = 17\,\text{kg}.$$

1.3.3 Addition of vectors by construction

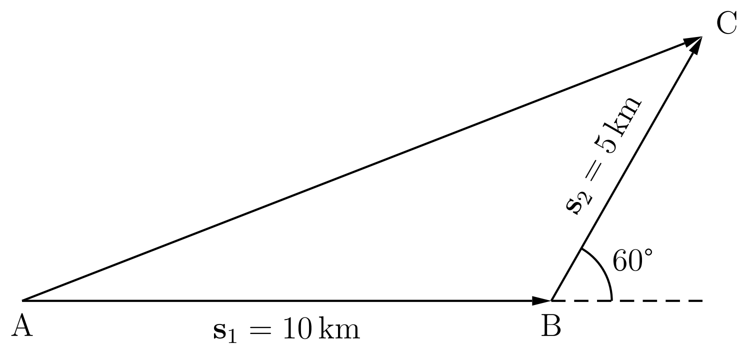
Vectors do **not** add arithmetically. One way of performing vector addition is with the aid of diagrams. When representing vector quantities in diagrams, arrows are used. It is understood that the **length of the arrow is proportional to the magnitude of the vector** and the **direction of the arrow indicates the direction of the vector**.

Example 4: Addition of vectors by construction

Suppose a car travels from point A in an easterly direction for 10 km to point B, and then travels another 5 km in a direction 60° north of east to point C. Determine the total distance travelled and the displacement of the car.

Solution:

The total distance travelled is 15 km, which is the sum of the distances travelled from A to B and from B to C.



To find the displacement, we first draw AB 10 units long representing the displacement in the easterly direction. From the end point of AB, we draw BC 5 units long at 60° north of east to represent the displacement in the north easterly direction. The total displacement (or **resultant**) is then represented by AC. The magnitude of the displacement is the length of AC whilst the direction is indicated by the arrow. Measurement shows that the length of AC is 13.3 km and the direction is 19.1° north of east.

Any number of vectors may be added together in a similar way. Consider the vectors shown in Figure 1a. To add these vectors, draw a scaled representation of any of the vectors (here the vector pointing west was chosen — the starting point is labelled 'P' in Figure 1b). Now take any of the other vectors and construct it from the endpoint of the previous vector. Continue in this way until all the vectors have been constructed. The resultant is then the vector drawn from the starting point to the end point of the last vector drawn (in this case the vector pointing south — the point labelled 'Q' in Figure 1b).

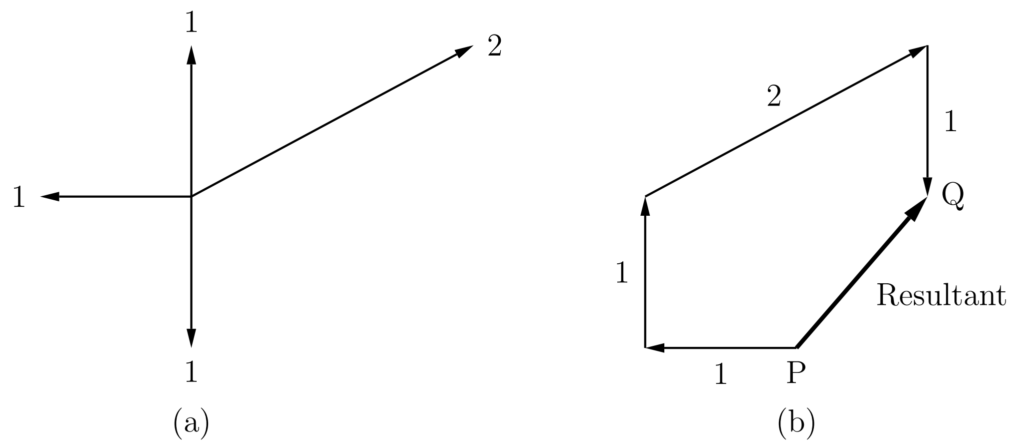


Figure 1: Adding several vectors by construction.

Note that, if in representing a number of vectors in a scale diagram the 'finishing' point **coincides** with the 'starting' point, then the resultant is zero. In such a case, if the vectors are forces acting on a body, then the body will be in **equilibrium** (see also Section 3.5.2).

1.3.4 Components of a vector

Finding the resultant of a number of vectors by construction is neither accurate nor convenient. In order to manipulate vectors algebraically, we need to introduce the concept of the **components** of a vector. An arbitrary vector $\mathbf{A}$ is shown in Figure 2. The x and y axes are drawn so that the origin coincides with the starting point of the vector $\mathbf{A}$. (In this course we will only consider vectors in two dimensions.) It is evident from Figure 2 that the vector $\mathbf{A}$ may be obtained by adding the vectors $\mathbf{A}_x$ and $\mathbf{A}_y$. $\mathbf{A}_x$ and $\mathbf{A}_y$ are known as the **vector components** of $\mathbf{A}$. The lengths of the components $\mathbf{A}_x$ and $\mathbf{A}_y$ are written A_x and A_y and are called the **scalar components** of $\mathbf{A}$. The scalar components A_x and A_y are positive if they point along the positive x and y axes respectively, and negative if they point along the negative x and y axes.

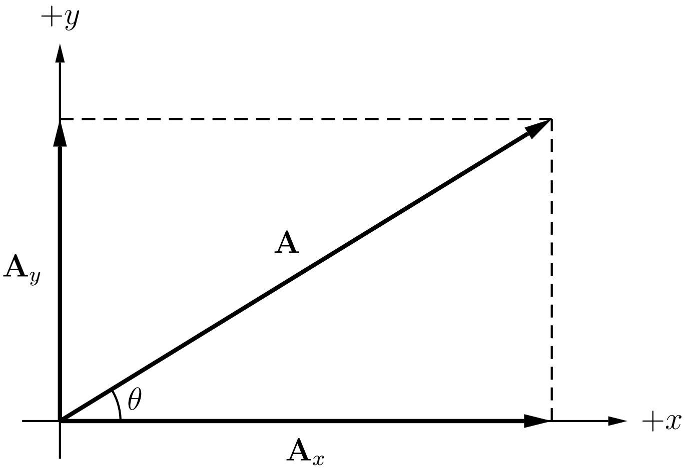


Figure 2: A vector $\mathbf{A}$and its vector components $\mathbf{A}_x$ and $\mathbf{A}_y$.

1.3.5 The resolution (or resolving) of vectors

If the magnitude and direction of a vector are known, we can find the components of the vector. This process is known as **resolving a vector into its components**. If the magnitude of the vector $\mathbf{A}$ in Figure 2 is A, and the direction of $\mathbf{A}$ is θ then

$$\boxed{A_x = A\cos\theta}, \tag{1}$$

and

$$\boxed{A_y = A\sin\theta}. \tag{2}$$

If the components A_x and A_y of a vector $\mathbf{A}$ are known, it is possible to find the magnitude and direction of $\mathbf{A}$. The magnitude may be found using the theorem of Pythagoras:

$$\boxed{A = \sqrt{A_x^2 + A_y^2}}. \tag{3}$$

The direction is found using the tangent, thus

$$\boxed{\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)}. \tag{4}$$

The values of A_x and A_y depend on the orientation of the axes (and indeed on the chosen coordinate system). The choice of orientation is purely a matter of convenience. In problems where it is necessary to resolve a vector into its components, a suitable choice of orientation can lead to substantial simplification of the solution. An important example is an object (weight **W**) at rest on an inclined plane (see Section 3.5.4)

1.3.6 The addition of vectors by means of components

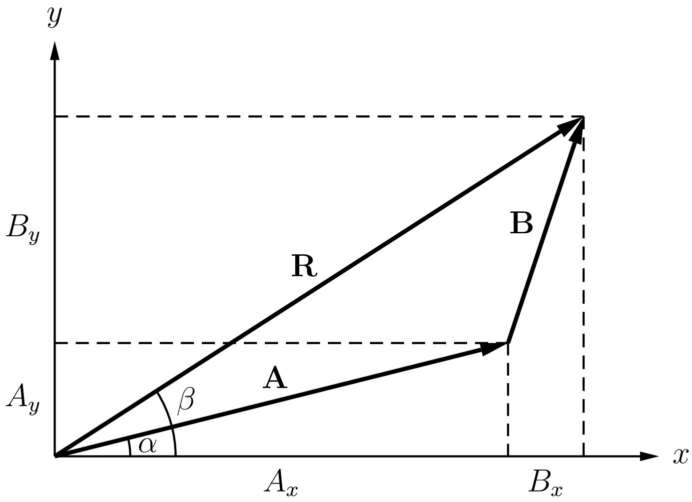


Figure 3: Addition of vectors using the components of the vectors.

Consider two vectors **A** and **B** (see Figure 3), we wish to find the resultant **R**. It should be obvious from figure 3 that the scalar components of **R** are given by

$$R_x = A_x + B_x$$

and

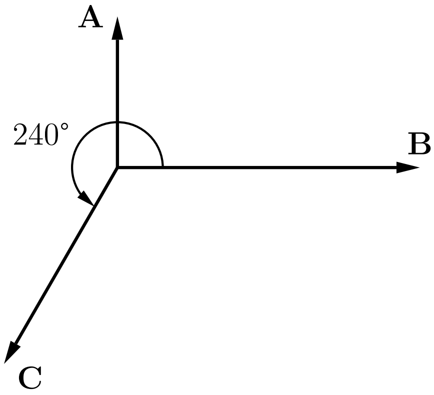
$$R_y = A_y + B_y,$$

where A_x, A_y, and B_x, B_y are the scalar components of **A** and **B** respectively.

The method of adding (or subtracting) vectors by adding (or subtracting) the components in this way may be extended to any number of vectors. We illustrate this by means of an example.

Example 5: Addition of vectors by adding components

Find the resultant of the three vectors in the drawing below. The vectors **A** and **B** are perpendicular to each other, and the magnitudes of **A**, **B** and **C** are 10, 20 and 15 units respectively.



Solution:

Using Equation (1) and (2) we first find the x and y components of each vector. We place the origin of the coordinate system at the intersection of the vectors, with the positive x axis along the vector **B**. Thus

$$A_x = 0 \quad \text{and} \quad A_y = 10, \qquad B_x = 20 \quad \text{and} \quad B_y = 0,$$

and

$$C_x = 15\cos 240^\circ = -7.5 \qquad \text{and} \qquad C_y = 15\sin 240^\circ = -13.$$

To find the resultant **R** we first calculate the scalar components of **R**:

$$\begin{aligned} R_x &= A_x + B_x + C_x = 0 + 20 - 7.5 = 12.5 \quad \text{and} \\ R_y &= A_y + B_y + C_y = 10 + 0 - 13 = -3. \end{aligned}$$

Using Equation (3) the magnitude of **R** is therefore

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{12.5^2 + (-3)^2} = 12.85$$

The direction of **R** is obtained from Equation (4):

$$\theta = \tan^{-1}\left(\frac{-3}{12.5}\right) = -13.5^\circ.$$

1.3.7 The parallelogram of vectors

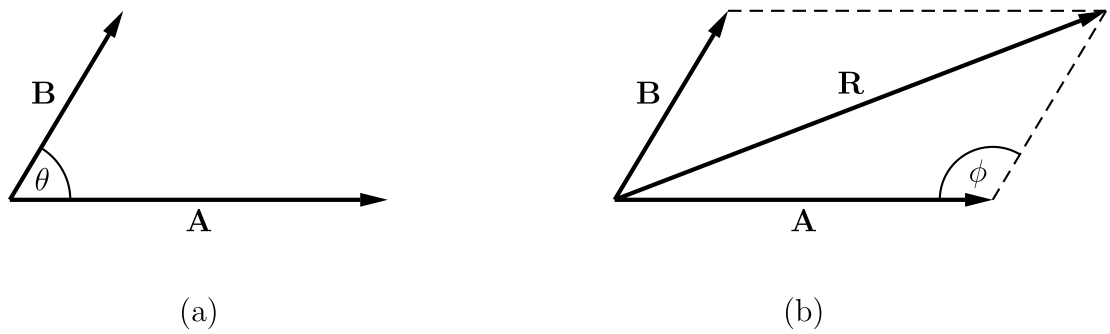


Figure 4: Parallelogram of vectors.

Consider two vectors **A** and **B** at an angle θ to each other as in Figure 4a. If we construct lines parallel and equal to **A** and **B**, we obtain a parallelogram with the resultant **R** as the diagonal (Figure 4b). The magnitudes of **A** and **B** and **R** are related by

$$R^2 = A^2 + B^2 - 2AB\cos\phi, \tag{5}$$

where $\phi = 180° - \theta$.

2 Kinematics

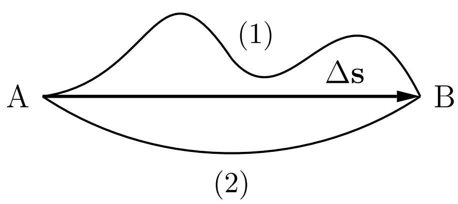
Kinematics is the study of the motion of bodies without reference to any forces. Forces will be considered in Section 3.

To describe the motion of an object, we need to specify where the object is with respect to some reference point. A **frame of reference** combines a reference point with a set of directions. The Cartesian coordinate system is a frame of reference, it consists of an origin and a set of mutually perpendicular directions, familiar to us as the x, y and z axes.

2.1 Definitions

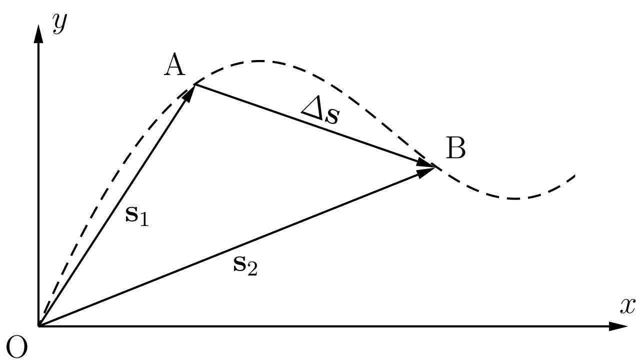
Displacement is the change in position of an object in a specified time interval.

Displacement is a vector quantity that has both magnitude and direction. In the diagram alongside, an object is displaced from A to B. The displacement is independent of the choice of path, thus the paths indicated by (1) and (2) correspond to the same displacement even though the distance covered in each case is clearly different.



Note that the distance an object moves is always positive, whereas the displacement $\Delta\mathbf{s}$ can be either positive or negative.

We now consider a particle which starts from the origin O and moves in the xy plane along the dotted path shown in the figure below.



At time t_1 the particle it is at A and at t_2 it is at B. In going from A to B, the particle's displacement is $\Delta\mathbf{s} = \mathbf{s}_2 - \mathbf{s}_1$. Let $\Delta t = t_2 - t_1$. This is the time taken to go from A to B.

The speed of an object is a measure of how fast it is moving.

Average speed is the total distance travelled divided by the time taken.

$$\text{average speed} = \frac{\text{distance AB along the dotted line}}{\Delta t}. \qquad (6)$$

The SI unit of speed is the metre per second ($\mathrm{m\,s^{-1}}$).

Uniform speed is achieved when equal distances are covered in equal times. When the distance travelled is plotted against the time taken, uniform speed produces a straight line graph, with the slope of the graph equal to the speed (see Figure 5).

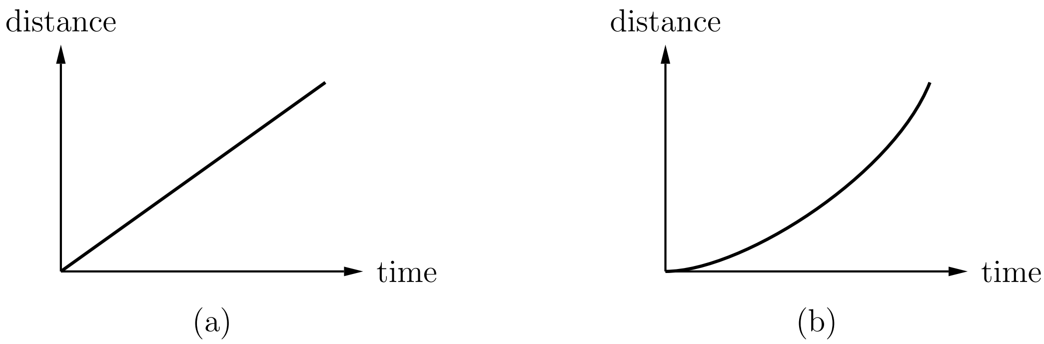


Figure 5: Graphs of distance vs time for (a) uniform speed and (b) non-uniform speed.

In order to describe the motion of an object, we need to know the direction in which it is moving as well as how fast it is moving. This information is given by the vector quantity **velocity**.

The **average velocity** in a given direction is the total displacement in that direction divided by the total time taken.

$$\Delta \mathbf{v} = \frac{\Delta \mathbf{s}}{\Delta t}, \qquad (7)$$

where $\Delta t = t_f - t_i$ is the total time.

Now suppose we let Δt become very 'small'. Then $\Delta \mathbf{s}$ is small and we obtain the instantaneous velocity.

The **instantaneous velocity** is the velocity at any instant during the motion of an object.

$$\mathbf{v} = \lim_{\Delta T \to 0} \frac{\Delta \mathbf{s}}{\Delta t} \qquad (8)$$

The acceleration of an object is a measure of how fast the velocity of an object is changing.

The **average acceleration** of an object in a given direction is the change in velocity divided by the total time taken.

Acceleration like velocity is a vector quantity:

$$\Delta \mathbf{a} = \frac{\Delta \mathbf{v}}{\Delta t}, \qquad (9)$$

where $\Delta \mathbf{v} = \mathbf{v}_f - \mathbf{v}_i$ and $\Delta t = t_f - t_i$.

The **instantaneous acceleration** is the acceleration at any instant during the motion of an object.

The instantaneous acceleration is defined in a similar way to the instantaneous velocity.

$$\mathbf{a} = \lim_{\Delta T \to 0} \frac{\Delta \mathbf{v}}{\Delta t} \tag{10}$$

Uniform acceleration is obtained when equal changes of velocity occur in equal time intervals. The graph of velocity versus time for uniform acceleration is a straight line (see Figure 6). The slope of the graph gives the magnitude of the acceleration.

It can be shown that **the area under the graph of velocity versus time is equal to the total displacement**. (See Example 9 and also Section 2.2.)

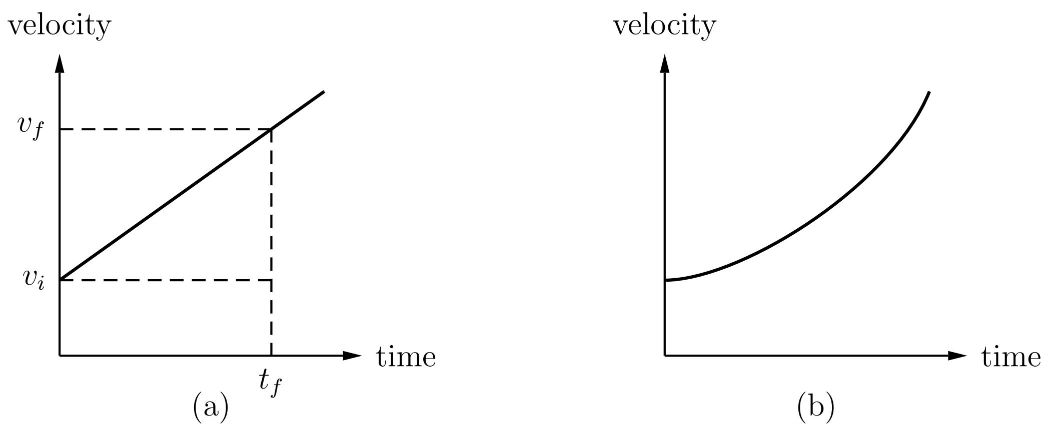


Figure 6: Graphs of velocity vs time in one dimension for (a) uniform acceleration and (b) non–uniform acceleration.

In future, when we speak about 'the velocity' or 'the acceleration' we will mean the instantaneous quantities.

Example 6: Velocity and speed

A jogger goes for her usual afternoon run. She leaves from her house and jogs a round trip of 10 km in 1 hour. What is her average speed and her average velocity after 1 hour?

Solution:

We determine the average speed from Equation (6). Hence

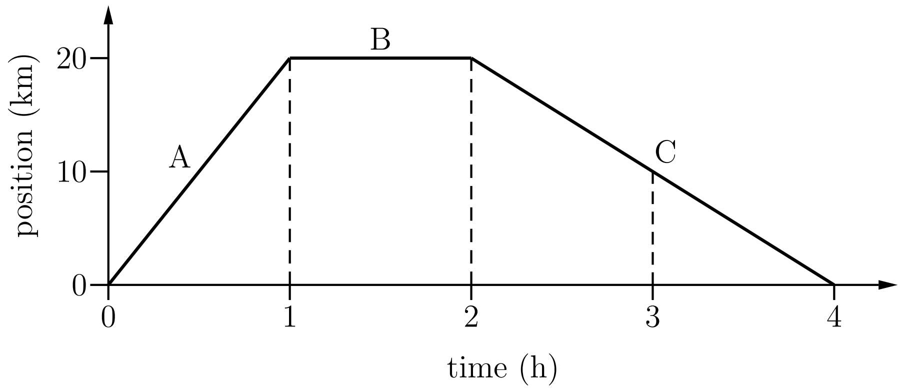
$$\text{average speed} = \frac{\text{distance}}{\text{time}} = \frac{10\,\text{km}}{1\,\text{h}} = 10\,\text{km}\,\text{h}^{-1}.$$

Since the jogger starts and finishes at the same point, the total displacement is zero. Hence the average velocity is also zero.

Example 7: Position-time graph

A cyclist rides at constant velocity and then stops for lunch. After lunch she rides back to the place from where she started, at a different constant velocity. Find the average velocity in each of the regions A, B and C indicated on the graph below.

Solution:



The displacement is the change in position. We may choose the coordinate system so that the position is along the x axis. In segment A of the journey, the initial position $\mathbf{s}_i = 0\,\hat{\mathbf{x}}\,\mathrm{km}$ and the final position is $\mathbf{s}_f = 20\,\hat{\mathbf{x}}\,\mathrm{km}$. The displacement $\Delta\mathbf{s} = 20-0 = 20\,\hat{\mathbf{x}}\,\mathrm{km}$. In segment B, $\mathbf{s}_i = \mathbf{s}_f$ so the displacement is zero, and in segment C, $\Delta\mathbf{s} = \mathbf{s}_f - \mathbf{s}_i = 0-20 = -20\,\hat{\mathbf{x}}\,\mathrm{km}$.

The average velocities may be found using Equation (7). Thus

$$\begin{array}{llllll} \text{Segment A} & \Delta\mathbf{v} & = \dfrac{\Delta\mathbf{s}}{\Delta t} & = \dfrac{20\,\mathrm{km}}{1\,\mathrm{h}} & = 20\,\hat{\mathbf{x}}\,\mathrm{km\,h^{-1}} \\ \text{Segment B} & \Delta\mathbf{v} & = \dfrac{\Delta\mathbf{s}}{\Delta t} & = \dfrac{0\,\mathrm{km}}{1\,\mathrm{h}} & = 0\,\hat{\mathbf{x}}\,\mathrm{km\,h^{-1}} \\ \text{Segment C} & \Delta\mathbf{v} & = \dfrac{\Delta\mathbf{s}}{\Delta t} & = \dfrac{-20\,\mathrm{km}}{2\,\mathrm{h}} & = -10\,\hat{\mathbf{x}}\,\mathrm{km\,h^{-1}} \end{array}$$

Example 8: Average acceleration

An aeroplane coming in to land is travelling at a speed of $100\,\mathrm{m\,s^{-1}}$. What is the average acceleration of the aeroplane if it comes to rest in a time of $10\,\mathrm{s}$?

Solution:

Assume the aeroplane is travelling in the positive-x direction. The average acceleration is given by Equation (9). The initial and final velocities are $\mathbf{v}_i = 100\,\hat{\mathbf{x}}\,\mathrm{m\,s^{-1}}$ and $\mathbf{v}_f = 0\,\hat{\mathbf{x}}\,\mathrm{m\,s^{-1}}$ corresponding to times $t_i = 0\,\mathrm{s}$ and $t_f = 10\,\mathrm{s}$. Thus

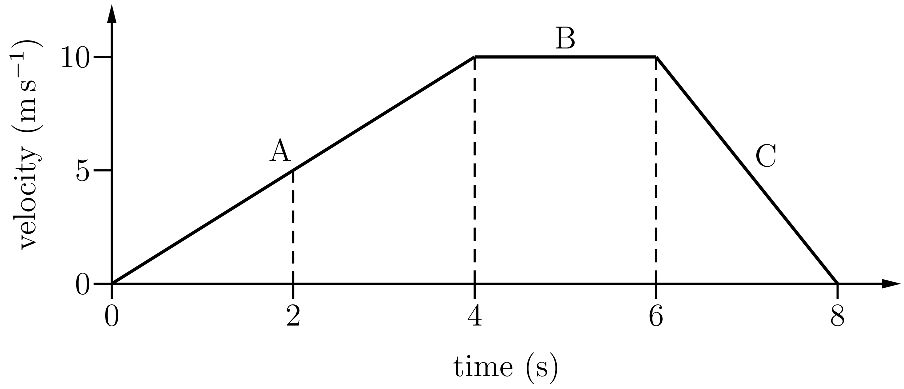
$$\Delta\mathbf{a} = \frac{\Delta\mathbf{v}}{\Delta t} = \frac{0-100}{10-0} = -10\,\hat{\mathbf{x}}\,\mathrm{m\,s^{-2}}.$$

Note that the acceleration is negative. This indicates that the acceleration is in the opposite direction to the velocity and that the aeroplane is slowing down. An object that slows down is said to decelerate or experience a retardation.

Example 9: Velocity-time graph

A car accelerates uniformly from rest, drives at constant velocity for a short while before decelerating, and coming to rest at a traffic light. Find the acceleration in each of the segments A, B and C indicated on the graph below. Find also the total displacement during segments A, B and C.

Solution:



We again choose the direction of motion along the x axis. The acceleration is given by the slope of the graph and the displacement by the area under the graph. In segment A the initial velocity $\mathbf{v}_i = 0\,\hat{\mathbf{x}}\,\mathrm{m\,s^{-1}}$ and the final velocity is $\mathbf{v}_f = 10\,\hat{\mathbf{x}}\,\mathrm{m\,s^{-1}}$. The change in velocity in segment A is therefore $10\,\mathrm{m\,s^{-1}}$. In segment B the slope of the graph is zero so the change in velocity is also zero, i.e. there is no acceleration. In the final segment, the change in velocity $\Delta\mathbf{v} = \mathbf{v}_f - \mathbf{v}_i = 0 - 10 = -10\,\hat{\mathbf{x}}\,\mathrm{m\,s^{-1}}$. The acceleration is in the opposite direction to the velocity, indicating that the car slows down.

The average acceleration in each segment may be found using Equation (9). Thus

$$\begin{array}{lllllll}
\text{Segment A} & \Delta\mathbf{a} & = & \dfrac{\Delta\mathbf{v}}{\Delta t} & = & \dfrac{10}{4} & = & 2.5\,\hat{\mathbf{x}}\,\mathrm{m\,s^{-2}} \\
\text{Segment B} & \Delta\mathbf{a} & = & \dfrac{\Delta\mathbf{v}}{\Delta t} & = & \dfrac{0}{2} & = & 0\,\hat{\mathbf{x}}\,\mathrm{m\,s^{-2}} \\
\text{Segment C} & \Delta\mathbf{a} & = & \dfrac{\Delta\mathbf{v}}{\Delta t} & = & \dfrac{-10}{2} & = & -5\,\hat{\mathbf{x}}\,\mathrm{m\,s^{-2}}
\end{array}$$

The displacement equals the area under the graph. We sum the area of triangle A, the rectangle B and the triangle C to obtain

$$\mathbf{s} = \tfrac{1}{2} \times 4 \times 10 + 2 \times 10 + \tfrac{1}{2} \times 2 \times 10 = 50\,\hat{\mathbf{x}}\,\mathrm{m}.$$

2.2 Kinematic equations for uniform acceleration in one dimension

In this section we derive the kinematic equations for an object travelling in a **straight** line with **uniform** acceleration. For motion in a straight line we drop the vector notation.

For convenience, we assume that the object is initially located at the origin. Thus $s_i = 0$ at $t = 0$ and after a time t, the object is located at s_f. To simplify the notation we write the elapsed time $t = \Delta t$, the displacement $s = \Delta s = s_f - s_i$, the initial velocity $u = v_i$ and the final velocity (after a time t), $v = v_f$.

For uniform acceleration, the average acceleration is the same as the instantaneous acceleration ($\Delta a = a$). From Equation (9) we obtain

$$a = \frac{\Delta v}{t} = \frac{v - u}{t}.$$

Rearranging this with v the subject, we find

$$\boxed{v = u + at.} \tag{11}$$

Variable	Symbol	Unit
time	t	s
displacement	s	m
initial velocity	u	$\mathrm{m\,s^{-1}}$
final velocity	v	$\mathrm{m\,s^{-1}}$
acceleration	a	$\mathrm{m\,s^{-2}}$

Table 4: Kinematic variables in one dimension.

The graph of Equation (11) is a straight line graph with slope a and intercept u (see Figure 6a).

From Equation (7) with $\Delta s = s$, the average velocity is

$$\bar{v} = \frac{s}{t}. \tag{12}$$

If the acceleration is constant, the velocity increases or decreases at a constant rate, the average velocity is therefore midway between the initial and final velocities. That is

$$\bar{v} = \tfrac{1}{2}(u + v). \tag{13}$$

Combining Equations (12) and (13) we find that

$$\boxed{s = \tfrac{1}{2}(u + v)t.} \tag{14}$$

Equation (14) represents the area under the graph of velocity versus time. (See Figure 6a and also Example 9.)

Substituting Equation (11) in Equation (14) we obtain

$$\boxed{s = ut + \tfrac{1}{2}at^2.} \tag{15}$$

Finally if we arrange Equation (11) with t as the subject and substitute this in Equation (14),

$$s = \tfrac{1}{2}(v + u)\left(\frac{v - u}{a}\right) = \frac{1}{2a}(v^2 - u^2),$$

and hence

$$\boxed{v^2 = u^2 + 2as.} \tag{16}$$

Example 10: Acceleration of a car from rest

A car accelerates uniformly from rest to $36\,\mathrm{km\,h^{-1}}$ in $4\,\mathrm{s}$. Calculate the magnitude of the acceleration and the distance covered during this $4\,\mathrm{s}$ interval.

Solution:

We have the following information:

$$\begin{aligned} u &= 0 \\ v &= 36\,\mathrm{km\,h^{-1}} = 10\,\mathrm{m\,s^{-1}} \\ t &= 4\,\mathrm{s} \\ a &= ? \\ s &= ? \end{aligned}$$

We can use Equation (11) to find the acceleration. Thus $v = u + at$ gives

$$a = \frac{v - u}{t} = \frac{10 - 0}{4} = 2.5\,\mathrm{m\,s^{-2}}.$$

The displacement may be found using Equation (15).

$$s = ut + \tfrac{1}{2}at^2 = 0 + \tfrac{1}{2} \times 2.5 \times 4^2 = 20\,\mathrm{m}.$$

Example 11: Distance covered by an accelerating spacecraft

A spacecraft is travelling with a velocity of $3000\,\mathrm{m\,s^{-1}}$ when it fires its retrorockets and begins to slow down with an acceleration whose magnitude is $10\,\mathrm{m\,s^{-2}}$. Determine the velocity of the spacecraft when its displacement is 200 km relative to the point at which the retrorockets were fired.

Solution:

Since the spacecraft slows down, the acceleration is in the opposite direction to the velocity. If we take the direction of motion of the spacecraft as the positive direction, then the acceleration is negative.

Data: $u = 3000\,\mathrm{m\,s^{-1}}$
$a = -10\,\mathrm{m\,s^{-2}}$
$s = 200\,000\,\mathrm{m}$
$v = ?$

We can use Equation (16) to find v. Hence

$$v^2 = u^2 + 2as = 3000^2 + 2 \times (-10) \times 200\,000 = 5\,000\,000\,\mathrm{m^2\,s^{-2}}.$$

The velocity is therefore $\pm 2236\,\mathrm{m\,s^{-1}}$. Both answers are acceptable: the negative value indicates a velocity in the opposite direction to the direction in which the spacecraft was initially moving. In other words, the retrorockets may have been fired long enough to slow the spacecraft to a halt and accelerate it in the opposite direction.

2.3 Motion under gravity

The acceleration g due to gravity, close to the earth's surface, is constant for all bodies at a given place in the absence of air resistance. The magnitude of the acceleration due to gravity on earth is approximately $g \simeq 9.8\,\mathrm{m\,s^{-2}}$.

For motion under gravity the equations of motions can be applied with the magnitude of the acceleration given by the value for g above. It must be remembered though that an object moving upwards suffers a **retardation** since the acceleration is in the **opposite** direction to the velocity.

Example 12: Motion under gravity

Calculate the time taken for a ball thrown vertically upwards, with an initial speed of $19.6\,\mathrm{m\,s^{-1}}$, to return to its starting point, neglecting air resistance.

Solution:

Since the ball returns to its starting position, the displacement is zero. If we take the upward direction as positive, we must use $a = -g = -9.8\,\mathrm{m\,s^{-2}}$. We can then use Equation (15) with the following values:

$s = 0$
$u = 19.6\,\mathrm{m\,s^{-1}}$
$a = -9.8\,\mathrm{m\,s^{-2}}$
$t = ?$

Thus $s = ut + \frac{1}{2}at^2$ gives $0 = ut + \frac{1}{2}at^2$, or

$$t = \frac{-2u}{a} = \frac{-2 \times 19.6}{-9.8} = 4\,\mathrm{s}.$$

We can also solve this problem treating the upward and downward motions separately. Consider first the upward motion. Here

$s = ?$
$u = 19.6\,\mathrm{m\,s^{-1}}$
$v = 0\,\mathrm{m\,s^{-1}}$
$a = -9.8\,\mathrm{m\,s^{-2}}$
$t = ?$

Using $v = u + at$ we find that for the upward motion $t = 2\,\mathrm{s}$. The displacement may now be calculated from $s = ut + \frac{1}{2}at^2$ giving

$$s = 19.6 \times 2 + \tfrac{1}{2} \times (-9.8) \times 2^2 = 19.6\,\mathrm{m}.$$

For the downward motion, we have

$s = -19.6\,\mathrm{m}$
$u = 0\,\mathrm{m\,s^{-1}}$
$v = ?$
$a = -9.8\,\mathrm{m\,s^{-2}}$
$t = ?$

We can use $s = ut + \frac{1}{2}at^2$ again with the above values to obtain

$$-19.6 = 0 + \tfrac{1}{2} \times (-9.8) \times t^2,$$

which gives $t = 2\,\mathrm{s}$. The total time taken is therefore $4\,\mathrm{s}$.

2.4 Strategies for problem solving

1. **Draw a diagram** to represent the situation. A simple 'block' drawing using arrows to indicate the directions of the various velocities and or accelerations is often all that is needed to gain a good understanding of a problem.

2. **Choose a set of coordinate axes** and decide which directions are to be called positive and negative. It is often convenient to place the origin at the place where an object starts its motion. In problems involving one dimension, the positive axis is usually chosen to go from left to right. For motion in two directions (see Section 2.5.1) the vertical direction is usually chosen as the y direction with 'up' being positive. To avoid confusion, **do not change your decision in the middle of a calculation**.

3. Write down the available values for the kinematic variables (s, u, v, a and t). Be careful to assign the appropriate sign depending on the choice of coordinate axes made in 2.

4. At least three of the kinematic variables should have values. Be sure to read the question carefully. There may be implied data like 'an object is accelerated from rest', in which case we may write $u = 0$.

5. Often a problem is divided in parts. For instance, a car may accelerate for a period of time, travel at constant velocity for a distance and then slow down. In such a case, divide the problem into parts, bearing in mind that the initial values for each part are given by the final values of the previous part.

2.5 Motion in a plane

2.5.1 Projectile motion

In this section we will consider the applications of the kinematic equations to the motion of projectiles.

If a body is projected at an angle to the vertical it travels along a curved path. At first sight it may therefore appear that the equations describing uniformly accelerated motion in a single line may not be applicable to this type of motion. The equations for straight line motion may be applied to the motion of a projectile **if the initial velocity is resolved into vertical and horizontal components**.

When the initial velocity is resolved into is vertical and horizontal components, we may treat these components independently. The **vertical motion** is treated as **uniformly accelerated motion in a straight line under gravity** and the **horizontal motion** is treated as **uniform motion (constant velocity) in which there is no acceleration**. At any time t in the projectile's motion, the actual velocity is then the **vector sum** of the separate horizontal and vertical velocities appropriate to the time t.

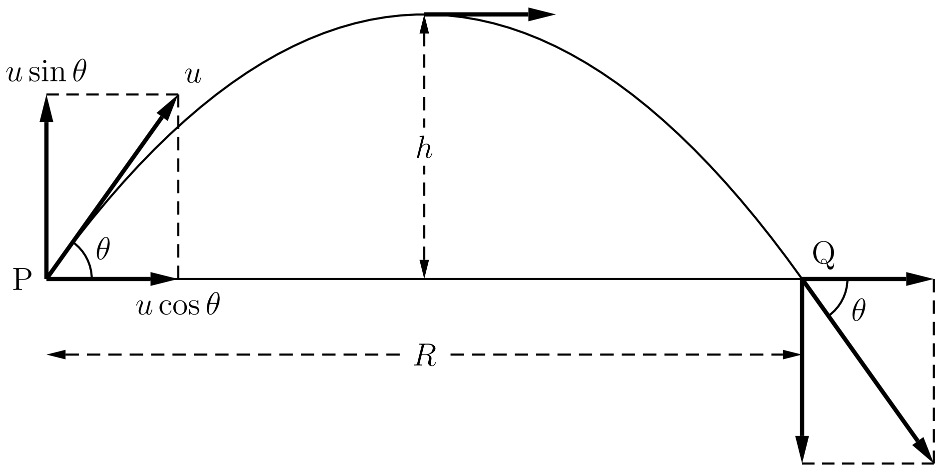


Figure 7: Motion of a projectile.

Figure 7 represents the motion of a projectile launched with an initial velocity u at an angle θ to the horizontal. We locate the origin of our coordinate system at the point P, with the positive x axis to the right and the positive y axis upwards. The vertical component of the velocity is $u_y = u\sin\theta$, the acceleration in the y direction is $a_y = -g$ and the net displacement in the y direction is zero since we assume the projectile is launched over horizontal ground.

We can determine the time of flight using Equation (15). Hence

$$s = ut + \tfrac{1}{2}at^2$$

which gives

$$0 = u_y t - \tfrac{1}{2}gt^2.$$

Using $u_y = u\sin\theta$ and rearranging, we find

$$t = \frac{2u\sin\theta}{g}\,. \tag{17}$$

The total distance covered in the horizontal direction is known as the **range** of the projectile (R in Figure 7). There is no acceleration in the horizontal direction ($a_x = 0$), hence the range is given by $s_x = u_x t$ or

$$R = u_x t = (u\cos\theta) \times \left(\frac{2u\sin\theta}{g}\right).$$

Using the trigonometric relation $\sin 2\theta = 2\sin\theta\cos\theta$, we obtain

$$R = \frac{u^2 \sin 2\theta}{g}. \tag{18}$$

The **maximum range** occurs when $\sin 2\theta = 1$, which gives $2\theta = 90°$ or $\theta = 45°$.

The height reached by the projectile may also be determined from Equation (15), using **half** the total time found in Equation (17). Thus

$$\begin{aligned} h = s_y &= u_y t + \tfrac{1}{2} a_y t^2 \\ &= (u\sin\theta) \times \left(\frac{u\sin\theta}{g}\right) - \frac{g}{2} \times \left(\frac{u\sin\theta}{g}\right)^2 \end{aligned}$$

which gives

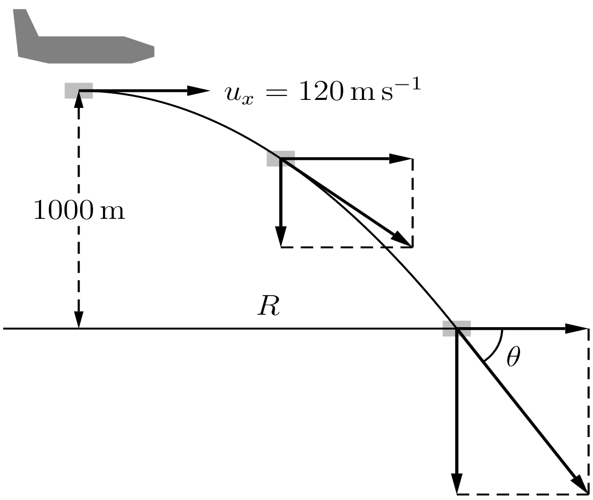
$$h = \frac{u^2 \sin^2\theta}{2g}. \tag{19}$$

Example 13: Projectile motion

An aeroplane flying 1000 m above level ground at120 m s^{-1} drops a relief package over a remote area. Ignoring air resistance, calculate

(a) how long the package takes to hit the ground,
(b) the horizontal and vertical components of the package's velocity, and hence
(c) the speed and the angle at which the package hits the ground.

Solution:



We take the origin of our coordinate system at the point where the package was released, the positive y axis points upwards and the positive x axis to the right.

(a) The time the package takes to fall to the ground depends only on the vertical distance the package must fall. In the y direction, the initial velocity $u_y = 0\,\mathrm{m\,s^{-1}}$, the displacement to the ground is $s_y = -1000\,\mathrm{m}$ and the acceleration is $a_y = -g = -9.8\,\mathrm{m\,s^{-2}}$. Putting $u = 0$ in Equation (15) gives $s = \frac{1}{2}at^2$, and hence

$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times (-1000)}{-9.8}} = 14.3\,\mathrm{s}.$$

(b) There is no acceleration in the horizontal direction (since we are ignoring air resistance), hence the horizontal velocity is $v_x = 120\,\mathrm{m\,s^{-1}}$. The vertical velocity increases in the negative y direction. Putting $u_y = 0$ in Equation (11) we have,

$$v_y = a_y t = -gt = -9.8 \times 14.3 = -140\,\mathrm{m\,s^{-1}}.$$

(c) The magnitude of the resultant velocity is obtained from Equation (3):

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{120^2 + (-140)^2} = 184\,\mathrm{m\,s^{-1}}.$$

The angle at which the package hits the ground is found using Equation (4):

$$\theta = \tan^{-1}\left(\frac{-140}{120}\right) = -49°.$$

2.5.2 Uniform circular motion

We now consider the problem of an object moving at constant speed v in a circle of radius r. Since the direction of the velocity is always changing, the object is, by definition, accelerating. It can be shown that the magnitude of this acceleration is

$$a = \frac{v^2}{r}, \tag{20}$$

and the direction is **towards the centre of the circle**.

Since this acceleration is constant in magnitude but not in direction, we **cannot** use the constant-acceleration equations for circular motion.

3 Dynamics

In Section 2 we studied kinematics. The motion of objects was described in terms of the observed quantities s, t, u, v, and a, but it was not considered what agent causes an object to move. The **dynamics** of an object is the **study of the motion of an object under the action of forces**.

We are familiar with the notions of force and mass from everyday usage. A force might be described as a 'push' or a 'pull' and mass as a measure of the size of an object, or the quantity of matter.

In the 17th century, Isaac Newton, building on the ideas of Galileo and others, developed laws and mathematical methods that enable us to define these concepts more rigorously, and treat the motions of objects under the action of forces in a mathematically consistent way.

3.1 Newton's first law of motion

Before Newton it was generally thought that a force was required to keep an object moving (see also Section 3.4.3). Newton instead postulated (on compelling experimental evidence) that an object would **continue** moving at constant speed in a straight line **unless** it was acted on by an unbalanced force.

Newton's first law of motion.
A body will continue in a state of rest, or of constant speed along a straight line, unless compelled by an **unbalanced** force to change that state.

Force is a **vector** quantity, and by the **unbalanced** or **net force** we mean the resultant force, or vector sum of all the forces acting on an object. Mathematically we write the net force

$$\boxed{\mathbf{F} = \sum \mathbf{F}_i}, \tag{21}$$

where $\sum$ is the mathematical symbol used to denote a sum of items indexed by the subscript i.

Newton's first law effectively sets the scene, because it defines the frame (or frames) of reference (i.e. coordinate system and clocks) with respect to which his remaining two laws of motion have meaning.

The class of reference frames with respect to which Newton's first law is valid are called **inertial reference frames**.

When a net force acts on an object, the object's velocity changes. The amount of change depends on the force as well as the mass of the object. If the same force acts on two objects of different mass, the more massive object will experience a smaller change in velocity. The tendency of an object with mass to resist a change in its state of motion is called the **inertia** of the object. The inertia of an object is measured by it's mass.

Inertia
Inertia is that property of a body by virtue of which it tends to persist in a state of rest or uniform motion in a straight line.

3.2 Newton's second law of motion

Newton's first law refers to a situation where there is no force. If a net force acts on an object, the object will change its state of motion according to Newton's second law.

Newton's second law of motion
If a net force acts on a body, the body will be accelerated; the magnitude of the acceleration is directly proportional to the magnitude of the net force and inversely proportional to the mass of the body, whilst the direction of the acceleration is in the direction of the net force.

In mathematical terms Newton's second law may be written as:

$$\mathbf{a} \propto \frac{\mathbf{F}}{m}$$

or

$$\mathbf{F} \propto m\mathbf{a},$$

where $\mathbf{a}$ is the magnitude of the acceleration of the mass m produced by the net force $\mathbf{F}$.

In SI units, force is defined so as to make the constant of proportionality equal to 1.

Unit of force

In the SI, the unit of force is the **newton** (N). The newton is defined as the net force which will give a mass of 1 kilogram an acceleration of $1\,\mathrm{m\,s^{-2}}$ in the direction of the force.

Using the above definition, Newton's second law takes the familiar form

$$\boxed{\mathbf{F}_{\mathrm{net}} = m\mathbf{a}} \tag{22}$$

where $\mathbf{F}$ is the **net force** in newtons (N), m in kilograms and $\mathbf{a}$ in $\mathrm{m\,s^{-2}}$.

3.2.1 Free-body diagrams

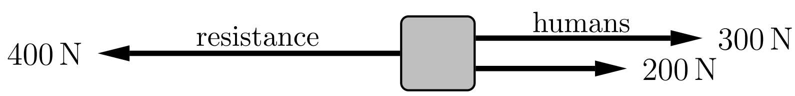
A **free-body diagram** is a diagram that represents an object and the forces acting on the object. The forces are represented by arrows (since force is a vector quantity) with length proportional to the magnitude of the force and the direction of the arrow indicating the direction of the force. A free-body diagram should **always** be drawn when a problem involves Newton's second law.

Example 14: Free-body diagram

Suppose two people push a car along a horizontal road. One person applies a force of 300 N and the other a force of 200 N. The car has a mass of 1200 kg and the total force due to resistance is 400 N and acts in the opposite direction to the forces exerted by the people pushing. Draw a free-body diagram that shows the horizontal forces on the car and find the acceleration of the car.

Solution:

We need to use Newton's second law. However we first need to find the resultant force. The free-body diagram below shows the car as a square and the arrows representing the forces.



Since the forces all act along one direction (the x direction say), the resultant force will also be in this direction, as will the acceleration. We choose the positive direction to the right, thus the net force is

$$F = \sum F_i = +300\,\mathrm{N} + 200\,\mathrm{N} - 400\,\mathrm{N} = +100\,\mathrm{N}.$$

The positive sign indicates that the force, **and hence the acceleration**, is in the direction we chose to be to the right.

The acceleration may now be determined from Equation (22):

$$a = \frac{F}{m} = \frac{100\,\mathrm{N}}{1200\,\mathrm{kg}} = 0.083\,\mathrm{m\,s^{-2}}.$$

3.3 Newton's third law

Newton's third law
If body A exerts a force on body B, body B exerts a force equal in magnitude and opposite in direction on body A.

Mathematically:

$$\mathbf{F}_{\mathrm{AB}} = -\mathbf{F}_{\mathrm{BA}} \tag{23}$$

If two people pull on spring scales hooked together, no matter how hard each person tries to pull, the readings on the two scales will be the same. Likewise, if a ball is hit with a bat, there is not only a force exerted by the bat on the ball, but also a force, which is the same in magnitude but opposite in direction, exerted by the ball on the bat. To walk, a person exerts a force on the earth. Consistent with Newton's third law, the earth exerts an oppositely directed force of equal magnitude on the person's foot and causes the person to move forward. Note that in Newton's third law and in all the above examples: **the two forces act on different bodies**.

3.4 Types of forces

There are four known types of forces in Nature.

1. **Gravitational force**
   Gravitation is the weakest of the known forces. It is also the only force that is purely attractive. Weight is a gravitational force (see Section 3.4.1).

2. **Electromagnetic force**
   Most forces we encounter in daily life are electromagnetic forces, they arise from the interaction of the electrically charged particles that make up atoms and molecules. Important examples are:

   - Normal contact forces like collisions and throwing of objects.
   - Tension forces such as surface tension and the tension in stretched strings.
   - Compressive forces such as those in springs or in a hydraulic press.
   - Frictional forces such as air drag on a skydiver and the grip on shoes (see Section 3.4.3).

   Many, if not most forces we encounter are combinations of forces.

3. **Weak nuclear force**
   A manifestation of the electromagnetic force that plays a role in the radioactive decay of atoms.

4. **Strong nuclear force**
   Plays a role in the interactions in nuclei of atoms.

3.4.1 The gravitational force and weight

Every particle in the universe attracts every other particle. The magnitude of the force acting on each of two particles of mass m_1 and m_2, separated by a distance r, is given by

$$\boxed{F = \frac{Gm_1m_2}{r^2}}, \tag{24}$$

where $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$ is the **universal gravitational constant**.

By a particle we understand something so small that it may be regarded as a mathematical point. Although Equation (24) is for 'point' particles, it can be used with good accuracy when the masses are small compared to the distance separating them. For objects that are not particles, r in Equation (24) is the distance between the **centres** of the objects.

The **weight** of an object is the gravitational force the earth exerts on it.

The weight always acts downward, towards the centre of the earth. An object will not necessarily weigh the same on another planet.

If the mass of an object is m and the acceleration due to gravity is g, then its weight on earth is given by

$$W = mg = \frac{GM_\mathrm{E}m}{R_\mathrm{E}^2},$$

where M_E and R_E are the mass and radius of the earth respectively. Here we used Newton's second law (Equation (22)) with the acceleration $a = g$. Since the mass m of the object appears on both sides of the equation, the acceleration due to gravity on a planet of mass M and radius R is given by

$$g = \frac{GM}{R^2}. \tag{25}$$

Example 15: The mass of the earth

Calculate the mass of the earth given that the radius of the earth is $R_\mathrm{E} = 6.38 \times 10^6\,\mathrm{m}$, $g = 9.8\,\mathrm{m\,s^{-2}}$ and $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$.

Solution:

We rearrange Equation (25) with M the subject. Then

$$\begin{aligned} M_\mathrm{E} = \frac{gR_\mathrm{E}^2}{G} &= \frac{9.8\,\mathrm{m\,s^{-2}} \times \left(6.38 \times 10^6\,\mathrm{m}\right)^2}{6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}} \\ &= 5.98 \times 10^{24}\,\mathrm{kg}. \end{aligned}$$

For distances greater than the earth's radius (as in the case of a satellite), we must add the height above the earth's surface to the radius.

Example 16: Acceleration due to gravity for a satellite in orbit

Determine the acceleration due to gravity for a satellite in orbit 200 km above the surface of the earth. Use the same data given in Example 15.

Solution:

We again use Equation (25) with the following data: $M = M_{\mathrm{E}} = 5.98 \times 10^{24}\,\mathrm{kg}$
$R = R_{\mathrm{E}} + h = (6.38 \times 10^{6} + 200 \times 10^{3})\,\mathrm{m}$ and
$G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$. Hence

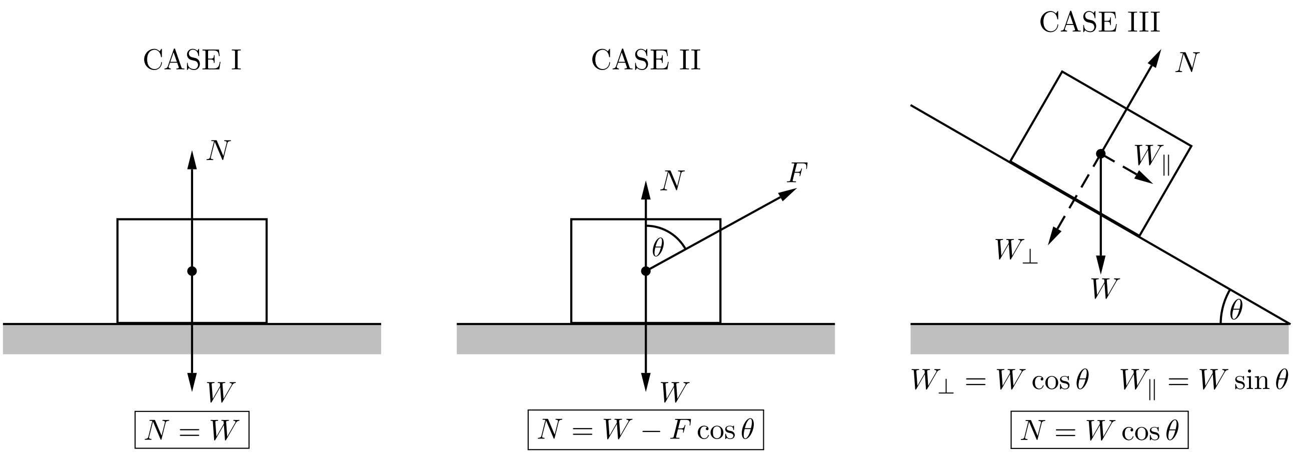
$$g = \frac{GM}{R^2} = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{(6.38 \times 10^{6} + 200 \times 10^{3})^2} = 9.2\,\mathrm{m\,s^{-2}}.$$

3.4.2 The Normal force

Normal in a mathematical sense means **perpendicular**. An object resting on a table for instance exerts a force equal to it's weight on the surface of the table. By Newton's third law, the table exerts an equal and opposite force on the object.

The **normal force** $\mathbf{F}_{\mathrm{N}}$ is the force, or component of a force, that a surface exerts on an object in contact with it.

The normal force N is often equal to the weight W of a body — but it is **not necessarily so**. Consider the following cases:



Example 17: Weight and the normal force in an elevator

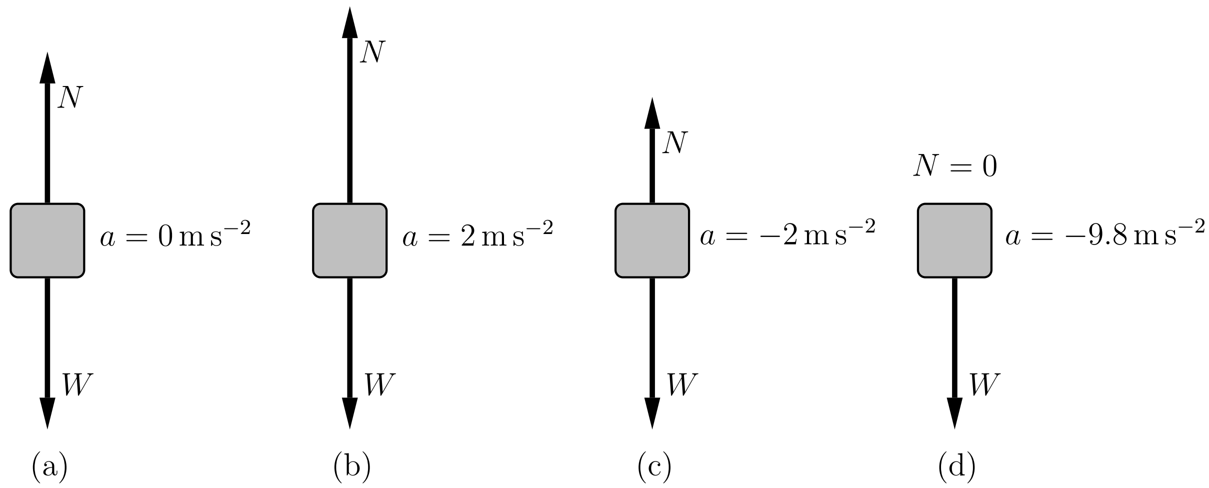
Find the apparent weight of a person whose mass is 60 kg in an elevator, when the elevator

(a) is stationary,

(b) accelerating upward at $2\,\mathrm{m\,s^{-2}}$,

(c) accelerating downward at $2\,\mathrm{m\,s^{-2}}$, and

(d) in free-fall.



Solution:

Imagine a person standing on a scale in the elevator. The apparent weight of the person is the normal force exerted by the scale on the man (the reading on the scale). Hence we must find the net force in each case and use Newton's second law with the acceleration a given by the acceleration of the lift. In the diagrams above, W is the weight of the person (unchanged in each case) and N the normal force. The net force on the person is the vector sum of the normal force and the weight. If we regard the upward direction as positive, then

$$F = -W + N,$$

where

$$W = mg = 60 \times 9.8 = 588\,\mathrm{N}.$$

Using Newton's second law ($F = ma$), the normal force is

$$N = ma + W.$$

(a) The acceleration is zero, hence

$$N = ma + W = 0 + 588 = 588\,\mathrm{N}.$$

(b) Here $a = 2\,\mathrm{m\,s^{-2}}$, therefore

$$N = ma + W = 60 \times 2 + 588 = 120 + 588 = 708\,\mathrm{N}.$$

(c) The acceleration is now $a = -2\,\mathrm{m\,s^{-2}}$, hence

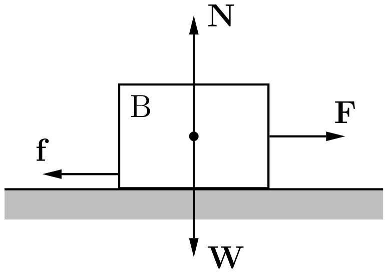
$$N = ma + W = 60 \times (-2) + 588 = -120 + 588 = 468\,\mathrm{N}.$$

(d) In free-fall, the lift (and person inside) is accelerating downwards at $9.8\,\mathrm{m\,s^{-2}}$. Thus $a = -9.8\,\mathrm{m\,s^{-2}}$ and

$$N = ma + W = 60 \times (-9.8) + 588 = -588 + 588 = 0\,\mathrm{N}.$$

3.4.3 Friction

The force that opposes the motion of one surface moving over another, with which it is in contact, is called the **force of friction**. Its magnitude depends on the materials of which the two surfaces are made, as well as on the force pressing them together. Some of the energy put into machines is transformed into heat energy because of the friction between moving parts. The heat may cause serious damage in addition to being wasteful. We cannot get rid of friction entirely but we can reduce it considerably by suitable choice of surfaces and by using lubrication.



Consider a force **F** applied to a block B on a horizontal surface S. If F is slowly increased from zero, the body remains at rest until F reaches a certain value, after which B accelerates in the direction of **F** (to the right in the diagram alongside).

As the force F increases, the **force of static friction** f_s increases and 'adjusts itself' to always exactly cancel F. The **maximum value** of the force of static friction is $f_s\,(\max)$. Experimentally we find that

$$\boxed{f_s\,(\max) = \mu_s N}\,, \tag{26}$$

where N is the normal force between the two surfaces in contact and μ_s, the **coefficient of static friction**, is a physical constant for the pair of surfaces in contact.

When the applied force F is larger than $\mu_s N$, there is a resultant force in the direction of **F** and the body will start to slip and accelerate in the direction of **F**. Once the block starts to slide, the frictional force drops below μ_s. We now talk about the **force of kinetic (or sliding) friction** f_k. Experimentally it is found that

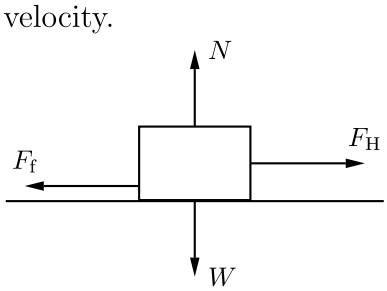
$$\boxed{f_k = \mu_k N}\,, \tag{27}$$

where μ_k, the coefficient of kinetic friction, is usually slightly less than μ_s. It is an experimental fact that μ_s depends only on the nature of the sliding surfaces; it is independent of the area of contact or the relative speed of the surfaces.

Example 18: Block on a horizontal surface with friction

A block which has a mass of 100 kg rests on a rough horizontal floor. The coefficient of sliding friction between the block and the floor is 0.25. Calculate the horizontal force F_H which would be required to move the block along the floor with constant velocity.

Solution:



When the block moves with constant velocity $a = 0$. Therefore, since $F = ma$, and $a = 0$, the total unbalanced force F must be zero. Taking the positive direction to the right,

$$F_H - F_f = 0.$$

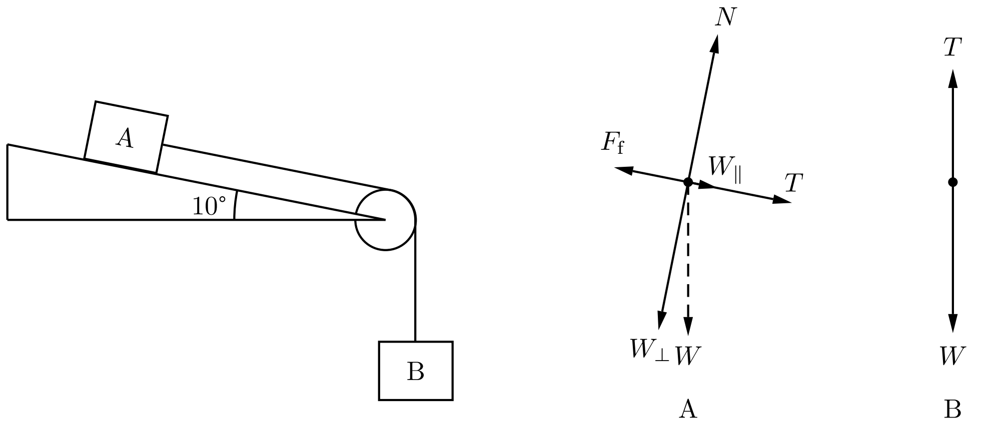
Also, $F_f = \mu N = mg\mu$, therefore

$$F_H = mg\mu = 100 \times 9.8 \times 0.25 = 245\,\text{N}.$$

Example 19: Block on an inclined plane with friction

A body of mass 20 kg rests on a plane surface AB inclined at 10° to the horizontal, B being lower than A. The mass is connected by a light string which passes over a pulley at B, to another mass of 20 kg that hangs freely below B. If the coefficient of sliding friction between the body and the surface of the plane is 0.40, calculate the acceleration with which these bodies would move, and the tension in the string connecting them.

Solution:



Consider the acceleration of both masses and choose the direction of motion as positive. The net force in the direction of motion is then

$$W - T + T + W_\parallel - F_\mathrm{f},$$

where the frictional force $F_\mathrm{f} = \mu N = \mu mg \cos 10°$. The total mass $m = 2m = 40\,\mathrm{kg}$. Newton's second law then gives

$$mg + mg \sin 10° - 0.4mg \cos 10° = 2ma$$

which gives $a = 3.82\,\mathrm{m\,s^{-2}}$. (Note that because the masses are equal, we do not need to know the mass for this calculation.)

The tension T can be obtained by considering the motion of the mass B. Newton's second law gives $W - T = ma$, hence

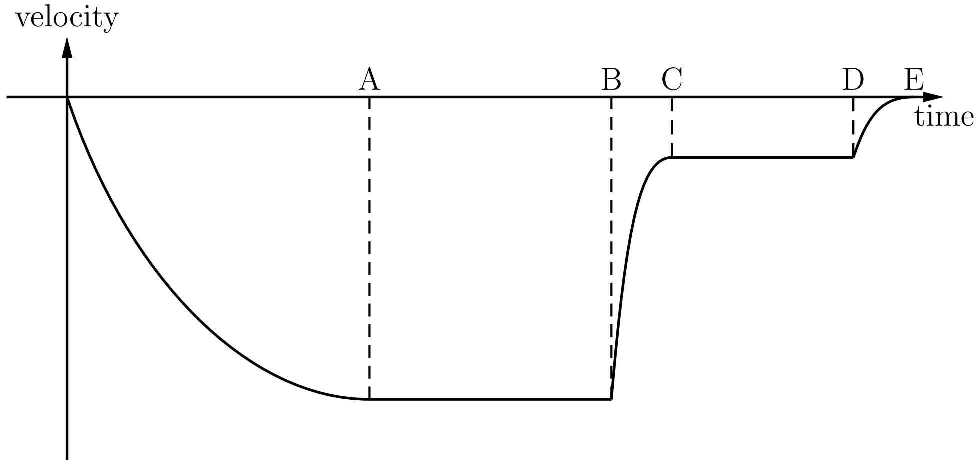
$$T = W - ma = 20 \times 9.8 - 20 \times 3.82 = 119.6\,\mathrm{N}.$$

A considerably smaller force called **rolling** friction is sufficient to keep one body moving against another if there are hard rollers or balls between two surfaces. It is important that the metal surfaces of roller or ball bearings that come into contact should be really hard. If one of the surfaces is not hard then the rolling friction might well be more than the sliding friction; it is for this reason that aircraft landing on soft snow fit skis in place of wheels.

An important example of Newton's first law is the case of an object falling through a medium. When it first starts to fall, it speeds up because its weight is bigger than the upthrust on it. But the dragging force on it increases as its velocity increases, and a stage can be reached when the upthrust plus dragging force (upwards) is as large as the weight (downwards). There is then no unbalanced force and the object continues to fall with the velocity it had reached — a constant velocity known as the **terminal velocity**.

Example 20: Velocity time graph for a skydiver

A skydiver jumps from an aeroplane. Sketch a velocity–time graph for the vertical motion of the skydiver indicating where the skydiver reaches terminal velocity, opens her parachute and lands.



Solution:

The origin represents the moment the skydiver jumps from the aeroplane. Her initial velocity in the vertical direction is zero at this point. Her acceleration on the other hand is equal to the acceleration due to gravity. Since we take the upward direction as positive, the acceleration is negative. (The slope of the graph gives the acceleration. If we draw a tangent to the curve at the origin, the slope of this line is negative). As the downward velocity increases, the **magnitude** of the acceleration decreases as the frictional drag due to the air increases. The force due to friction is in the **opposite** direction to the velocity. At A the skydiver reaches terminal velocity — here the force due to the air resistance is equal and opposite to the force due to gravity. The slope of the graph between A and B is zero and the velocity remains the same. At B she opens her parachute. Her acceleration here is positive and is a maximum as she opens her parachute, decreasing to zero as she again reaches terminal velocity at C. The segment between D and E is where she reaches the ground.

3.4.4 Tension

If two people pull on either end of a rope there will be a certain tension in the rope. The force experienced by each person will be the same and will equal the tension in the rope. Figure 8 depicts the free-body diagram for this scenario. Each end of the rope provides the reaction

Figure 8: Tension in a rope

force on the person pulling at that end, as required by Newton's third law. The force of the people pulling on the rope in effect gets transmitted through the rope.

3.5 Application of Newton's laws

In this section we discuss applications of Newton's laws to various systems. In Section 3.5.2 we consider systems that are in equilibrium and in Section 3.5.3 some examples of non-equilibrium situations are discussed.

3.5.1 Guidelines for solving problems involving Newton's laws

The following points must be remembered when the relationship $F = ma$ is used:

1. The mass in Newton's second law ($F = ma$) represents the **total mass** accelerated by the **net force**.
2. F represents the **net force in the direction of motion**. If the accelerated mass is acted upon by a number of forces, the total net component of the forces in the direction of the motion must be calculated.

The following systematic approach to problems in which the relationship $F = ma$ has to be used may be useful:

1. Draw a diagram representing the **general** situation.
2. Select **one object** from the situation whose motion is to be analysed and draw a **free-body diagram** for this object. For this, the object is removed from its environment, together with **all the forces** exerted on it by bodies with which it interacts.
3. Select a convenient origin and orientation of the coordinate axes.
4. Write an expression for the **net force**.
5. Apply Newton's second law.

3.5.2 Equilibrium applications

Equilibrium
An object is in equilibrium when it has zero acceleration.

When the acceleration of an object is zero, the net force on the object is zero by Newton's first law. Thus when an object is in equilibrium in two dimensions, we must have

$$\sum F_{\mathrm{x}} = 0 \tag{28a}$$
$$\sum F_{\mathrm{y}} = 0. \tag{28b}$$

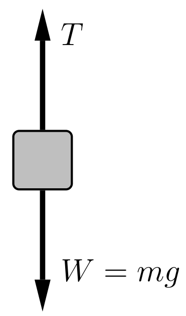
Examples of systems in equilibrium include a book lying on a table, a lamp hanging from a cord or a vehicle moving at constant velocity.

Example 21: Tension in a cord, one dimension (equilibrium case)

A lamp is suspended from the ceiling by a cord. If the lamp has a mass of 5 kg, determine the tension in the cord.
Solution:

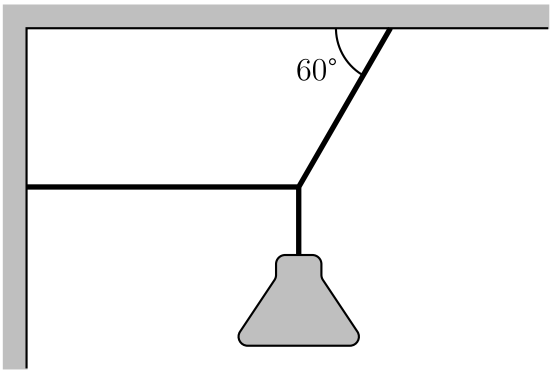
Since the system is in equilibrium, we use Equation (28b) to find the net force. $T - W = 0$ gives

$$T = W = mg = 5\,\mathrm{kg} \times 9.8\,\mathrm{m\,s^{-2}} = 49\,\mathrm{N}.$$



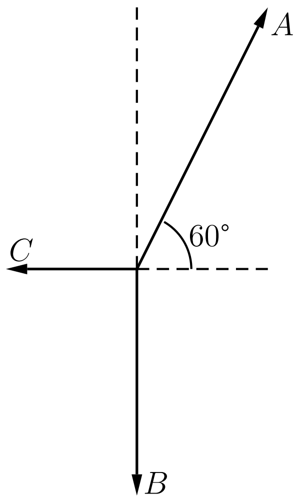
Example 22: Tension in a cord, two dimensions (equilibrium case)

A lamp is suspended by three cords as depicted in the diagram below. The cord attached to the ceiling makes an angle of 60° with the ceiling and the cord attached to the wall is stretched horizontally. If the lamp has a mass of 5 kg, determine the tensions in the cords.



Solution:

Since the forces (tensions in the cords) do not act in the same direction, we will need to resolve the components of the forces in the x and y directions. First we construct a free-body diagram representing the forces acting at the intersection of the cords (the magnitudes of the forces are labelled A, B and C).



The tension B is simply equal to the weight of the lamp. Hence

$$B = W = mg = 5 \times 9.8 = 49\,\text{N}.$$

Equating the x and y components of A, B and C according to Equations (21), we have

$$A_x + B_x + C_x = 0 \quad \text{and} \quad A_y + B_y + C_y = 0,$$

where

$$\begin{aligned} A_x &= A\cos 60^\circ & \text{and} \quad A_y &= A\sin 60^\circ, \\ B_x &= 0 & \text{and} \quad B_y &= -B, \\ C_x &= -C & \text{and} \quad C_y &= 0. \end{aligned}$$

Thus, equating first the y components:

$$A\sin 60^\circ - 49 + 0 = 0, \quad \text{which gives} \quad A = 56.6\,\text{N}.$$

Equating the x components:

$$A\cos 60^\circ + 0 - C = 0, \quad \text{which gives} \quad C = 28.3\,\text{N}.$$

3.5.3 Non-equilibrium applications

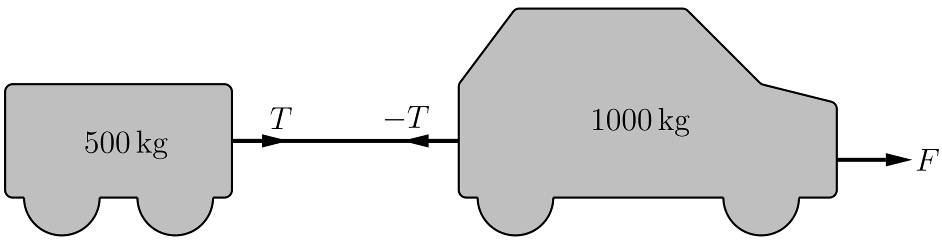
When an object is not in equilibrium, there are unbalanced forces acting on the object and hence the net force is non-zero. The approach to solving non-equilibrium problems is almost identical to the approach used to treat equilibrium problems. Instead of equating the net force to zero as in Equations (28), we must use Newton's second law. Thus for an accelerating object in two dimensions

$$\sum F_{\mathrm{x}} = ma_x \tag{29a}$$

$$\sum F_{\mathrm{y}} = ma_y. \tag{29b}$$

Example 23: The tension in a rope (non-equilibrium)

Suppose the magnitude of the net force accelerating a car and trailer is $F = 3000\,\mathrm{N}$. The mass of the car is 1000 kg and the mass of the trailer is 500 kg. Determine the acceleration of the car and trailer, and the tension in the rope. Assume the mass of the rope is negligible.



Solution:

The acceleration may be determined by applying Newton's second law to the whole system (the car and trailer). We are given the net force and we know the total mass of the system. Thus

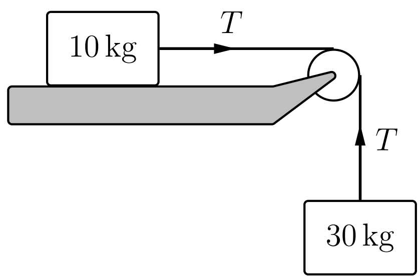
$$a = \frac{F}{m} = \frac{3000}{1000 + 500} = 2\,\mathrm{m\,s^{-2}}.$$

Newton's second law may also be applied to the trailer by itself. Here the net force in the horizontal direction is the tension T and the mass of the system is the mass of the trailer. The acceleration was found above. Hence

$$F = T = ma = 500 \times 2 = 1000\,\mathrm{N}.$$

Example 24: Objects connected by a rope

A block of mass 10 kg on a table is attached to a block of mass 30 kg by a rope passing over a pulley as shown in the diagram alongside. Ignoring all frictional effects and assuming the pulley to be massless, find (a) the acceleration of the two blocks and (b) the tension in the cord. (Take $g = 10\,\mathrm{m\,s^{-2}}$.)



Solution:

(a) The net force available to accelerate the system is due to the 30 kg mass (the forces on the 10 kg block in the y direction are equal and opposite). Hence

$$F = W = -mg = -30\,\text{kg} \times 10\,\text{m}\,\text{s}^{-2} = -300\,\text{N}.$$

The total mass of the system is the mass of the two blocks, $m_{\text{TOT}} = 10\,\text{kg} + 30\,\text{kg} = 40\,\text{kg}$. We can now use Newton's second law to find the acceleration.

$$a = \frac{F}{m} = \frac{-300\,\text{N}}{40\,\text{kg}} = -7.5\,\text{m}\,\text{s}^{-2}.$$

(b) the only unbalanced force on the 10 kg mass is due to the tension T in the rope. Using Newton's second law with $a = 7.5\,\text{m}\,\text{s}^{-2}$, since the 10 kg block is accelerated in the positive x direction,

$$T = ma = 10\,\text{kg} \times 7.5\,\text{m}\,\text{s}^{-2} = 75\,\text{N}.$$

3.5.4 Motion on a smooth inclined plane

When a block of mass m is placed on a smooth frictionless inclined plane, as shown in Figure 9, the block will be accelerated down the plane. The force in the direction of motion which gives rise to the acceleration of the mass is the component of the weight of the body down the plane. The weight W of the body (which is a force acting vertically downwards) can be resolved into components acting along the plane and perpendicular to the plane (see also Section 1.3.5).

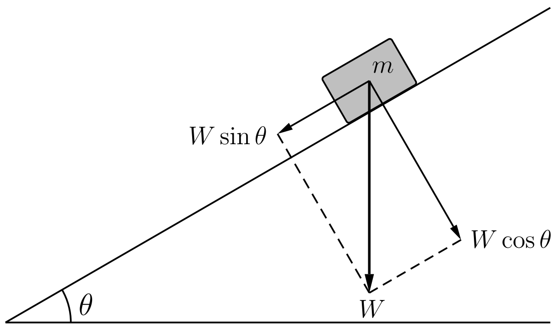


Figure 9: An object on a smooth inclined plane.

If the plane is inclined at an angle θ to the horizontal, then the component of the weight parallel to the plane is

$$W_{\parallel} = W \sin\theta,$$

and the component perpendicular to the plane is

$$W_{\perp} = W \cos\theta.$$

Example 25: Motion on an inclined plane

Show that the acceleration of a sliding body down a frictionless plane is **independent of the mass of the body**.

Solution:

Consider a body with mass m which is placed on a frictionless plate inclined at an angle θ to the horizontal as in Figure 9. The net force in the direction of the motion is

$$F_{\|} = W \sin\theta.$$

The total mass accelerated is m. Using Newton's second law, we have

$$F = ma = F_{\|} = W \sin\theta = mg \sin\theta,$$

and hence

$$a = g \sin\theta,$$

which is independent of the mass m.

3.6 The centripetal force

For an object moving at constant speed v in a circle of radius r the centripetal acceleration is given by

$$a = \frac{v^2}{r}$$

(see Section 2.5.2). By Newton's first law, if an object is accelerating, there must be a force acting on it. For circular motion this force is known as the **centripetal force**. The centripetal force may take different forms. For instance, for a car travelling in a circular path on a horizontal surface the centripetal force is the frictional force between the tyres and the surface; for a satellite in orbit around a planet the centripetal force is the gravitational force; for an object at the end of a string spun in a circular path, the tension in the string provides the centripetal force. Newton's second law takes the form

$$\sum F_{\text{net}} = m\frac{v^2}{r}, \tag{30}$$

where the resultant force causing the circular motion $\sum F_{\text{net}}$ is called the centripetal force. The centripetal force is directed towards the centre of rotation (in the same direction as the acceleration).

3.7 Satellites in circular orbits

Consider an object of mass m (e.g. a satellite) moving in a circular orbit (radius r) at a constant speed v_0 around the earth. Newton's law of gravitation, the second law $F = ma$ with $a = v_0^2/r$ (see Equations (24), (22) and (20)), then gives

$$\frac{GmM}{r^2} = m\left(\frac{v_0^2}{r}\right),$$

or

$$v_0^2 = \frac{GM}{r}. \tag{31}$$

The period T of this object is the time to complete one full orbit.

$$\text{Time} = \text{circumference}/\text{speed, so}$$
$$T = \frac{2\pi r}{v_0}. \tag{32}$$

Squaring (32) and substituting from (31) gives

$$T^2 = \frac{4\pi^2 r^2}{v_0^2} = \frac{4\pi^2 r^2}{GM/r}$$

or

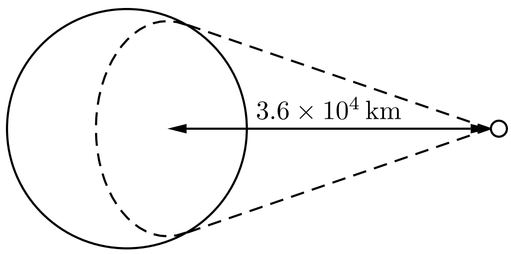
$$T^2 = \frac{4\pi^2 r^3}{GM}.$$

This important result, which shows that the square of the satellite's orbit is proportional to the cube of its orbital radius, is known as **Kepler's third law**. You should remember that

$$\boxed{T^2 \propto r^3}. \tag{33}$$

3.7.1 Geostationary orbits

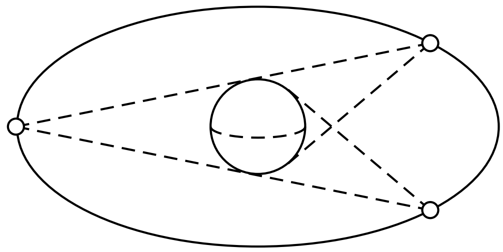
As long ago as 1945 the scientist and science fiction writer Arthur C. Clarke had suggested that, if a satellite were placed above the equator at a height such that its orbital period was equal to the rotational period of the Earth, it would appear stationary from a point on the Earth's surface. This characteristic would enable the satellite to provide permanent coverage of a given area.



A geostationary orbit

By substituting the value of T as 8.64×10^4 s (i.e. 42 hours) the value of R is found to be 4.23×10^7 m. Taking the Earth's radius as 6.37×10^6 m, the height of the orbit is 3.59×10^7 m or 3.6×10^4 km. This is called a **geostationary orbit**.

Clarke had further suggested that if three such satellites were equally spaced in geostationary positions above the equator then communication coverage of most of the world would be possible except for the polar regions.



World-wide coverage with geostationary satellites

4 Hydrostatics

The science of hydrostatics is the study of fluids at rest. For our purposes we take a fluid to be a liquid or a gas.

4.1 Density

The **density** ρ of a substance is its mass per unit volume.

$$\boxed{\rho = \frac{m}{V}}. \tag{34}$$

The density of a substance changes with both temperature and pressure. Therefore when the density of a substance is given, the temperature should also be given.

The density of pure water at 4 °C is 1000 kg per cubic metre, i.e. 1000 kg m^{-3} (or 1 g cm^{-3}).

4.2 Relative density

The relative density (RD) of a substance is defined as the ratio of the density of the substance and the density of water. Thus

$$\boxed{\mathrm{RD} = \frac{\text{density of substance}}{\text{density of water (at same temp.)}}}. \tag{35}$$

Relative density is a ratio and therefore has **no units**.

Note that since density = mass/volume, then if we consider equal volumes of the substance and water, the expression for RD becomes

$$\mathrm{RD} = \frac{\text{mass of a given substance}}{\text{mass of an equal volume of water (at same temp.)}}.$$

Relative density is sometimes known as the **specific gravity**. It may be determined for both solids and liquids — see Practical Manual.

Example 26: Relative density of aluminium.

The density of aluminium is 2700 kg m^{-3}. Find the relative density of aluminium given that the density of water is 1000 kg m^{-3}.

Solution:

We can apply Equation (35) directly. Hence

$$\mathrm{RD} = \frac{2700\,\mathrm{kg\,m^{-3}}}{1000\,\mathrm{kg\,m^{-3}}} = 2.7.$$

Example 27: Relative density of a mixture.

10 cm^3 of a liquid A whose relative density is 0.8 is mixed with 15 cm^3 of a liquid B whose RD is 1.2. If there is no contraction on mixing, find the relative density of the mixture.

Solution:

To calculate the RD of the mixture, we need to find the density of the mixture. We are given the volume of each liquid, thus we first find the mass of each liquid.

Suppose the density of water is ρ, and let the density and mass of liquid A and B be ρ_{A}, m_{A} and ρ_{B}, m_{B} respectively. For 10 cm^3 of A

$$\mathrm{RD} = \frac{\rho_{\mathrm{A}}}{\rho} = \frac{m_{\mathrm{A}}/10}{\rho} = 0.8,$$

which gives

$$m_{\mathrm{A}} = 8\rho.$$

Similarly for liquid B

$$\mathrm{RD} = \frac{\rho_\mathrm{B}}{\rho} = \frac{m_\mathrm{B}/15}{\rho} = 1.2,$$

which gives

$$m_\mathrm{B} = 18\rho.$$

The density of the liquid is therefore

$$\rho_\mathrm{AB} = \frac{m_\mathrm{A} + m_\mathrm{B}}{10 + 15} = \frac{26\rho}{25}.$$

Finally, we can determine the density of the mixture:

$$\mathrm{RD} = \frac{\rho_\mathrm{AB}}{\rho} = \frac{26\rho}{25\rho} = 1.04.$$

4.3 Pressure

Pressure
If a force F acts over an area A **perpendicular** to the force, then the **pressure** P is the force per unit area.

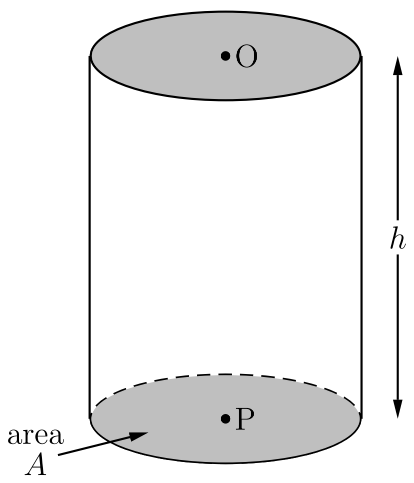
$$\boxed{P = \frac{F}{A}}. \tag{36}$$

The SI unit of pressure is the $\mathrm{N\,m^{-2}}$ or the pascal ($1\,\mathrm{Pa} = 1\,\mathrm{N\,m^{-2}}$).

4.3.1 Summary of some laws of pressure in fluids at rest

A fluid is a liquid or a gas.

1. The pressure at a depth h in a fluid at **rest**, due to the fluid itself, is $h\rho g$ pascals where ρ is the density of the fluid.



Let O be a point in the surface of a fluid and let P be a point in the fluid a distance h vertically below O. Imagine a cylinder of cross-sectional area A having OP as its axis, as in the diagram. The whole weight of the cylinder of fluid acts on the base around P.

$$\begin{aligned}\text{Weight of cylinder} &= \text{mass} \times g \\ &= \text{volume} \times \text{density} \times g \\ &= (A \times h) \times \rho \times g\end{aligned}$$

Substituting the above expression for the weight in Equation (36), we find

$$P = \frac{(A \times h) \times \rho \times g}{A},$$

which gives

$$P = h\rho g. \tag{37}$$

The pressure $h\rho g$ is sometimes called the 'hydrostatic pressure'. So the **absolute pressure** at P equals (the pressure at the surface O) + (the hydrostatic pressure). If the pressure at O is the atmospheric pressure P_0, then

$$\boxed{P = P_0 + h\rho g}. \tag{38}$$

In other words, in a fluid at rest, the pressure increases linearly with distance below the surface, assuming that the density of the liquid remains constant throughout.

2. At any point in a liquid which is at rest the **total pressure** is the pressure on the **surface** of the liquid **plus the pressure due to the liquid** itself

3. At any two points in the **same horizontal** plane in any one liquid which is at rest, the pressures are the **same**. (Otherwise the liquid would flow.)

4. Pressure applied to the surface of a liquid is transmitted equally throughout the liquid in every direction. (This is called **Pascal's Law**.) This principle is used in the **Hydraulic Press** (or the Bramah Press – after Joseph Bramah (1748–1814), locksmith and inventor).

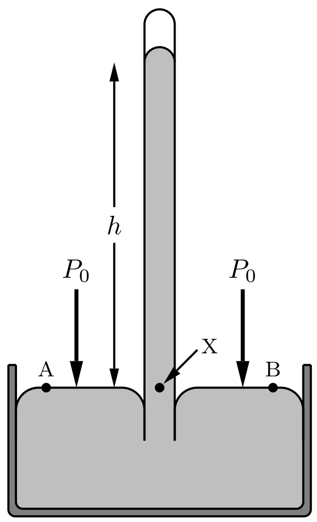
4.3.2 Gauge pressure

As the name implies, this is the pressure recorded by a pressure gauge, and is frequently the difference between absolute pressure and atmospheric pressure. The equation $\Delta P = h\rho g$ indicates why it is convenient to refer to pressures by **heads of liquid**.

A unit commonly used for gas pressures is the atmosphere (atm), which is defined to be 101 325 Pa. It is essentially equivalent to that exerted by 760 mm of mercury (mmHg) of specified density under standard gravity. Note that 1 mmHg exerts a pressure of 133 Pa.

Pressure can be measured by various means. Two ways are described below.

The simple barometer



Atmospheric pressure P_0 'balances' the pressure due to the mercury column of height h. The pressure at A or B therefore equals the pressure at X. Hence

$$P_0 = h\rho g,$$

where $\rho = 13\,600\,\mathrm{kg\,m^{-3}}$ is the density of mercury. Standard atmospheric pressure corresponds to a mercury height of 0.76 m, i.e.

$$\begin{aligned} P_0 &= 0.76 \times 13\,600 \times 9.8 \\ &= 1.01 \times 10^5\,\mathrm{N\,m^{-2}}. \end{aligned}$$

This pressure is often stated simply as '76 centimetres of mercury'.

Another unit of pressure often used is the bar, it is related to pascals in the following way:

$$\begin{array}{lclclcl} 1\,\text{bar} & \equiv & 10^5\,\text{N}\,\text{m}^{-2} & = & 10^5\,\text{Pa} & = & 100\,\text{kPa} \\ 1\,\text{mbar} & = & 10^{-3}\,\text{bar} & \equiv & 10^2\,\text{N}\,\text{m}^{-2} & = & 100\,\text{Pa}. \end{array}$$

Garage pressure gauges read tyre pressure in bars **in excess** of atmospheric pressure.

Standard atmospheric pressure is approximately 1013 millibars (mbar). Near the ground, atmospheric pressure decreases by about 1 cm of mercury per 120 m above sea level. Thus, in Pietermaritzburg (about 600 m above sea level), atmospheric pressures are usually about 71 cmHg, or 950 mbar.

Example 28: A water barometer.

Calculate the height of a water barometer corresponding to standard atmospheric pressure.

Solution:

We need to calculate the height of a column of water that will give a pressure of $1.01 \times 10^5\,\text{N}\,\text{m}^{-2}$. Using Equation (37) with $\rho = 1000\,\text{kg}\,\text{m}^{-3}$:

$$h = \frac{P_0}{\rho g} = \frac{1.01 \times 10^5\,\text{N}\,\text{m}^{-2}}{1000\,\text{kg}\,\text{m}^{-3} \times 9.8\,\text{m}\,\text{s}^{-2}} = 10.3\,\text{m}.$$

Example 29: Pressure due to a column of air.

If the average density of air in the science block is $1.20\,\text{kg}\,\text{m}^{-3}$ and a barometer reads 71.00 cmHg at ground level, what will it read on the roof 30 m up? Take the density of mercury as $13\,600\,\text{kg}\,\text{m}^{-3}$.

Solution:

The Pressure due to 30 m of air is given by Equation (37):

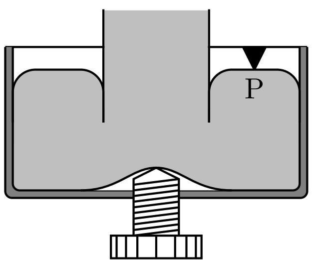
$$P = h\rho g = 30 \times 1.20 \times 9.8 = 353\,\text{N}\,\text{m}^{-2}.$$

The height of mercury (H) required to give this pressure is

$$H = \frac{P}{\rho g} = \frac{353}{13\,600 \times 9.8} = 0.0026\,\text{m} = 0.26\,\text{cm}.$$

The barometer on the roof will therefore read $71.00 - 0.26 = 70.74$ cmHg.

The Fortin barometer

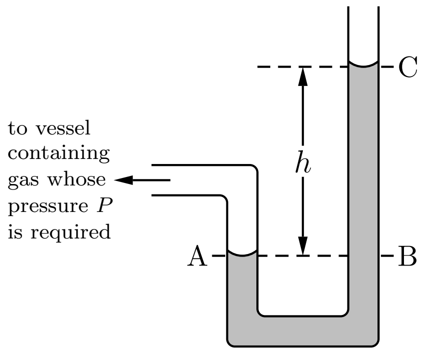


The Fortin barometer is an accurate barometer and differs in two important ways from a simple barometer:

1 The mercury level in the reservoir is adjusted so as to touch the tip, P, of a pointer which is at the zero of the height scale.

2 The height of the mercury column is read very accurately using a 'vernier' scale. (See the Fortin barometer in the laboratory.)

The manometer



The pressure (P) at A is equal to the pressure at B (same level). But the pressure at B is equal to atmospheric pressure P_0 plus the pressure due to the column h, of fluid.

$$\begin{aligned}\text{Pressure at A} &= \text{Pressure at B}\\ \text{Pressure at B} &= \text{Atmospheric pressure at B} + h\rho g\\ \therefore\quad P &= P_0 + h\rho g.\end{aligned}$$

(If C is below the level of A, $P = P_0 - h\rho g$.)

4.4 Archimedes' principle

Archimedes' Principle

When a body is wholly or partly immersed in a fluid, it experiences an upthrust or apparent loss of weight, equal to the weight of fluid displaced.

This principle is true for any solid displacing any fluid (liquid or gas).

In the case of a floating body the full weight of the body is supported by the upthrust of the fluid in which it is floating. This application of Archimedes' principle is called the **law of flotation**, and may be stated as follows:

Law of flotation

A floating body displaces its own weight of the fluid in which it floats.

Example 30: Apparent weight of a mass suspended in a liquid.

A copper ball (relative density 8.9) has a mass of 267 g. Calculate (a) its weight in air (b) its apparent weight when suspended in water.

Solution:

(a) The weight in air, $W = mg = 267 \times 10^{-3}\,\text{kg} \times 9.8\,\text{m}\,\text{s}^{-2} = 2.62\,\text{N}$.

(b) By Archimedes' principle, the ball experiences an upthrust equal to the weight of water displaced. Hence the weight when suspended in water is equal to the weight in air, minus the upthrust it experiences due to the water. We can determine the mass of water displaced (and hence the upthrust) from Equation (35) considering equal volumes:

$$\text{RD} = \frac{\text{mass of ball}}{\text{mass of water displaced by ball}},$$

which gives

$$\text{mass of water displaced by ball} = \frac{267 \times 10^{-3}\,\text{kg}}{8.9} = 0.030\,\text{kg}.$$

The upthrust is equal to the weight of water displaced, which is $F_{\text{upthrust}} = mg = 0.030\,\text{kg} \times 9.8\,\text{m}\,\text{s}^{-2} = 0.294\,\text{N}$. The apparent weight of the ball when suspended in water is therefore

$$W_{\text{apparent}} = W - F_{\text{upthrust}} = 2.62 - 0.294 = 2.326\,\text{N}.$$

5 Work, energy and power

Work and energy are concepts we use every day for any number of different things. For the purposes of formal study however, we need precise definitions that enable us to interpret the concepts of work and energy in a consistent way. From these definitions we deduce relations that are applicable to real systems.

5.1 The work done by a constant force

We consider a **constant** force of magnitude F acting on an object of mass m at an angle θ as shown in Figure 10. The force moves the object over a distance s.

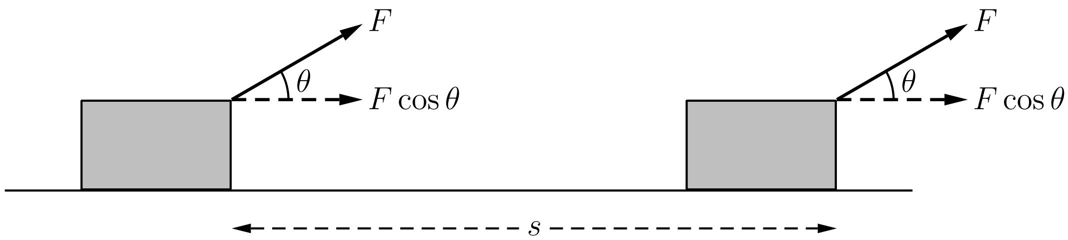


Figure 10: An object moved a distance s by a force F.

The amount of **work** done on an object by a force is equal to the product of the displacement and the **component** of the force in the **direction** of the displacement.

$$\boxed{W = Fs\cos\theta}\,, \tag{39}$$

where F and s are the magnitudes of $\mathbf{F}$ and $\mathbf{s}$. The angle θ is the included angle between $\mathbf{F}$ and $\mathbf{s}$. Note that:

1. If the applied force is in the direction of the displacement, then $\theta = 0°$ and $W = Fs$.
2. If the applied force is in the opposite direction of the displacement, then $\theta = 180°$ and $W = -Fs$.
3. If the applied force is perpendicular to the displacement, then $\theta = 90°$ and $W = 0$. (i.e. a force acting at right angles to a displacement does **no** work.)

Work is a scalar quantity and the unit of work is the newton-metre or joule.

One **joule** is the work done by a force of one newton when it moves its point of application through a distance of one metre in the direction of the force ($1\,\text{joule} \equiv 1\,\text{J} \equiv 1\,\text{N m}$).

Example 31: Work done on an object dragged over a distance

Find the work done when a trunk is dragged a distance of 10 m by a force of 50 N applied at an angle of 45° above the surface over which the trunk is moved.

Solution:

The work done may be obtained directly from Equation (39) with the values given. Hence

$$W = Fs\cos\theta = 50\,\text{N} \times 10\,\text{m} \times \cos 45° = 354\,\text{J}.$$

5.2 Energy

Different forms of energy are identified:

(a) Kinetic energy

(b) Potential energy

- gravitational
- elastic
- electrostatic

(c) Thermal and internal energy

(d) Radiant energy

(e) Chemical energy

(f) Nuclear energy

(g) Mass energy

On a microscopic scale, **all** forms of energy can be classified as either (a) or (b).

Changes occur between different forms of energy, and the amounts possessed by different bodies, but if we take all forms into account, we find there is no change in the total energy in the universe. This is the **law of conservation of energy**. Mathematically:

$$\boxed{\begin{array}{lcl}\text{Total energy of a closed system} & = & \text{Total energy of a closed system} \\ \text{before some event} & & \text{after the event}\end{array}}$$

5.2.1 Kinetic energy and the work-energy theorem

Suppose a constant force F acts on an object of mass m. If the object moves a distance s in the direction of the force F, we can obtain the work done on the object by multiplying $F = ma$ on both sides by the displacement:

$$W = Fs = mas.$$

Since the force acting on the object is constant, the acceleration of the object is also constant and we may apply the kinematic equations of motion for constant acceleration. Substituting $v^2 = u^2 + 2as$ (with as the subject) in the equation above and using v_i and v_f for the initial and final velocities instead of u and v, we obtain

$$W = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2. \tag{40}$$

If the **kinetic energy** of an object is defined as

$$\boxed{E_\mathrm{k} = \tfrac{1}{2}mv^2}, \tag{41}$$

then the right hand side of Equation (40) represents the change in kinetic energy of the object when an amount of work W is done on it. Although we have derived Equation (40) for the work done by a constant force, it can be shown to hold in general for the work done by any type of force. Furthermore, if more than one force does work on an object, the total work done is equal to the work done by the **resultant** force:

$$\boxed{W(\text{due to the resultant force}) = \Delta E_\mathrm{k} = E_\mathrm{k}(\text{final}) - E_\mathrm{k}(\text{initial})}. \tag{42}$$

Equation (42) is known as the **work-energy theorem** for an object. The work done on an object can be positive or negative depending on the size of the angle θ in Equation (39).

Note that the kinetic energy E_k is a *positive scalar quantity* that represents the energy associated with a body because of its motion. It is either

1. the work done *by* the resultant force in accelerating the body from rest to an instantaneous speed v, or
2. the work done *by* the body on some external agent which brings it to rest.

(1) and (2) are equivalent.

The work done on an object by the resultant force is the same as the sum of the work done by each force separately. Different types of energy are associate with the work done by different types of forces.

Example 32: The work done in accelerating a car

A 1000 kg car accelerates uniformly from rest to a speed of $30\,\mathrm{m\,s^{-1}}$ in a distance of 20 m. Determine

(a) the kinetic energy gained,
(b) the work done by the net force acting on the car, and
(c) the magnitude of the average net force.

Solution:

(a) The car starts from rest, so the initial kinetic energy is zero and the kinetic energy gained is the final kinetic energy. From Equation (41):

$$E_\mathrm{k} = \tfrac{1}{2}mv^2 = \tfrac{1}{2} \times 1000 \times 30^2 = 4.5 \times 10^5\,\mathrm{J}.$$

(b) By the work-energy theorem, the work done is equal to the kinetic energy gained, hence $W = 4.5 \times 10^5\,\mathrm{J}$.

(c) From Equation (39) with $\theta = 0$ (since we must assume the force accelerating the car acts in the direction of the displacement of the car),

$$F = \frac{W}{s} = \frac{4.5 \times 10^5\,\mathrm{J}}{20\,\mathrm{m}} = 2.25 \times 10^4\,\mathrm{N}.$$

5.2.2 Potential energy

Potential energy is the energy possessed by a system by virtue of the relative positions of its component parts

Gravitational potential energy

Suppose we exert forces on a body of mass m and on the earth, and thereby push the body m to a rest position a vertical distance h above its initial position. The work $W = Fs = mgh$ is done against the gravitational force. We say that the system has gained gravitational potential energy

$$\boxed{E_\mathrm{p} = mgh}\,. \tag{43}$$

There is no gain in kinetic energy. The pulls of the earth on the body and the body on the earth have done *negative work*.

When the system is released the two gravitational forces both do positive work on the body and on the earth. Both, in principle, acquire kinetic energy, but that gained by the earth is negligible. The potential energy is associated with the relative positions (i.e. separation) of the two masses making up the system.

Gravitational potential energy is a kind of energy that can be completely recovered and converted into kinetic energy.

Example 33: Work done in lifting an object

Find the work done in lifting a body whose mass is 5 kg through a vertical distance of 2 m.

Solution:

From Equation (43):

$$W = mgh = 5\,\mathrm{kg} \times 9.8\,\mathrm{m\,s^{-2}} \times 2\,\mathrm{m} = 98\,\mathrm{J}.$$

Elastic potential energy (stretching/compressing a spring)

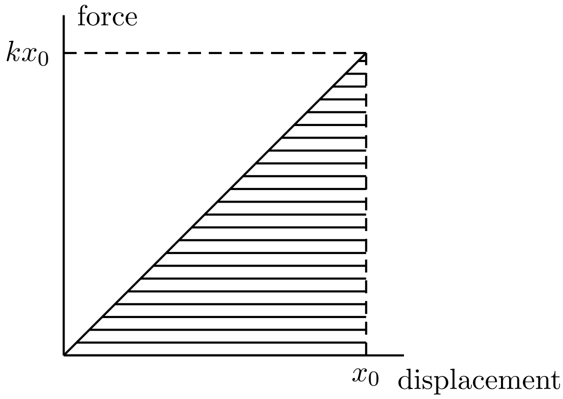
Consider a spring having natural length ℓ_0. Suppose the spring is stretched by an amount x to a new length ℓ (i.e. $x = \ell - \ell_0$). **Hooke's law** gives us the force F exerted by the spring. It is

$$\boxed{F = -kx}, \tag{44}$$

where the force constant k depends on the spring. The minus sign indicates that F points in the opposite direction to the displacement. This is a **restoring force**.

We cannot use $W = Fs$ to determine the work done in stretching this spring because F **is not constant**. It depends on the extension x.

To find the work done in stretching a spring by an amount x_0, we consider the graph alongside. This graph shows that the force varies linearly with x. It is **not** constant.



The work done is the area under a force–displacement graph. The shaded area is $W = \frac{1}{2} \times \text{base} \times \text{height}$.

$$W = \tfrac{1}{2} \times x_0 \times kx_0 = \tfrac{1}{2}kx_0^2.$$

This is the work done by an external agent in stretching the spring. This work is stored as elastic potential energy in the spring until we release the spring.

The potential energy for a spring is given by

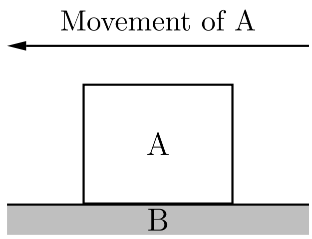
$$E_\mathrm{p} = \tfrac{1}{2}kx^2.$$

Note that x is the extension/compression from the spring's natural length.

Internal energy

A frictional force always opposes relative motion. When surfaces slide over one another, such a force always does negative work. This work represents energy being transferred to random molecular potential and kinetic energy (**internal energy**).

In the figure alongside, B exerts a frictional force on A to the right, which moves its point of application to the left, and so does negative work. Macroscopically we see that A experiences a force which reduces its speed. Microscopically work is being done on a molecular scale that results in an increase of the random kinetic and potential energies of individual molecules. We observe a temperature increase along the common surface.



5.3 Conservation of mechanical energy

The mechanical energy of an object is defined as the sum of its potential and kinetic energies:

$$E = E_\mathrm{p} + E_\mathrm{k}. \tag{45}$$

If there is no work done on an object by any applied forces, then the **mechanical energy** of the object is **conserved**. This means that the total mechanical energy of the object always remains the same. Hence $\Delta E = 0$ and

$$\Delta E_\mathrm{p} + \Delta E_\mathrm{k} = 0. \tag{46}$$

Equation (46) may be rewritten in terms of the final and initial kinetic and potential energies in the useful form

$$(E_\mathrm{p} + E_\mathrm{k})_\mathrm{final} = (E_\mathrm{p} + E_\mathrm{k})_\mathrm{initial}. \tag{47}$$

Example 34: Conservation of mechanical energy

A mass of 80 kg slides down a smooth inclined plane 16 m high and 80 m long. Neglecting friction,

(a) calculate the potential energy of the mass at the top of the slope.

(b) How much kinetic energy does it have at the bottom of the slope?

(c) Determine the speed of the mass at the bottom of the slope.

Solution:

(a) Relative to the bottom of the slope, the potential energy at the top is

$$\Delta E_\mathrm{p} = mgh = 80\,\mathrm{kg} \times 9.8\,\mathrm{m\,s^{-2}} \times 16\,\mathrm{m} = 1.25 \times 10^4\,\mathrm{J}.$$

(b) Mechanical energy is conserved, hence the potential energy lost equals the kinetic energy gained. The kinetic energy at the bottom of the slope is therefore $E_\mathrm{k} = 1.25 \times 10^4\,\mathrm{J}$.

(c) Rearranging Equation (41):

$$v = \sqrt{\frac{2 \times E_\mathrm{k}}{m}} = \sqrt{\frac{2 \times 1.25 \times 10^4\,\mathrm{J}}{80\,\mathrm{kg}}} = 17.7\,\mathrm{m\,s^{-1}}.$$

5.4 Power

Power is the **rate** at which work is done.

If work W is done in a time t, then the average power P for the time interval t is given by

$$\boxed{P = \frac{\text{work}}{\text{time}} = \frac{W}{t}}. \tag{48}$$

Power is not associated with any direction, and since work and time are scalar quantities, power is also a scalar quantity. The SI unit of power is the watt (W).

One **watt** is the power developed when one joule of work is done per second.

If the force doing work is in the same direction as the displacement, then Equation (39) becomes

$$W = Fs. \tag{49}$$

Substituting Equation (49) in Equation (48), we obtain the useful expression

$$P = \frac{W}{t} = \frac{Fs}{t} = F\bar{v}. \tag{50}$$

The velocity in Equation (50) is the **average velocity**, and the force is in the direction of the motion.

Example 35: The power generated by an accelerating car

A car whose mass is 1000 kg accelerates constantly from rest at $2.0\,\mathrm{m\,s^{-2}}$ for 10 s. Determine the average power generated by the net force accelerating the car.

Solution:

We first find the force accelerating the car. Using Equation (22), we have

$$F = ma = 1000\,\mathrm{kg} \times 2\,\mathrm{m\,s^{-2}} = 2000\,\mathrm{N}.$$

To find the power, we must either calculate the work done and use Equation (48), or the average velocity and use Equation (50). We will demonstrate both methods.

Method 1: To find the work done, we need to determine the distance travelled. Using Equation (15), the displacement

$$s = ut + \tfrac{1}{2}at^2 = 0 + \frac{2\,\mathrm{m\,s^{-2}} \times (10\,\mathrm{s})^2}{2} = 100\,\mathrm{m}.$$

The work done is therefore

$$W = Fs = 2000\,\mathrm{N} \times 100\,\mathrm{m} = 2 \times 10^5\,\mathrm{J},$$

and the power generated is

$$P = \frac{W}{t} = \frac{2 \times 10^5\,\mathrm{J}}{10\,\mathrm{s}} = 2 \times 10^4\,\mathrm{W}.$$

Method 2: Since the acceleration is constant, the average velocity is half the initial plus final velocity. As the initial velocity is zero, we have

$$\bar{v} = \tfrac{1}{2}v.$$

Then from Equation (11):

$$\bar{v} = \tfrac{1}{2}v = \tfrac{1}{2}(u + at) = \tfrac{1}{2}(0 + 2\,\mathrm{m\,s^{-2}} \times 10\,\mathrm{s}) = 10\,\mathrm{m\,s^{-1}}.$$

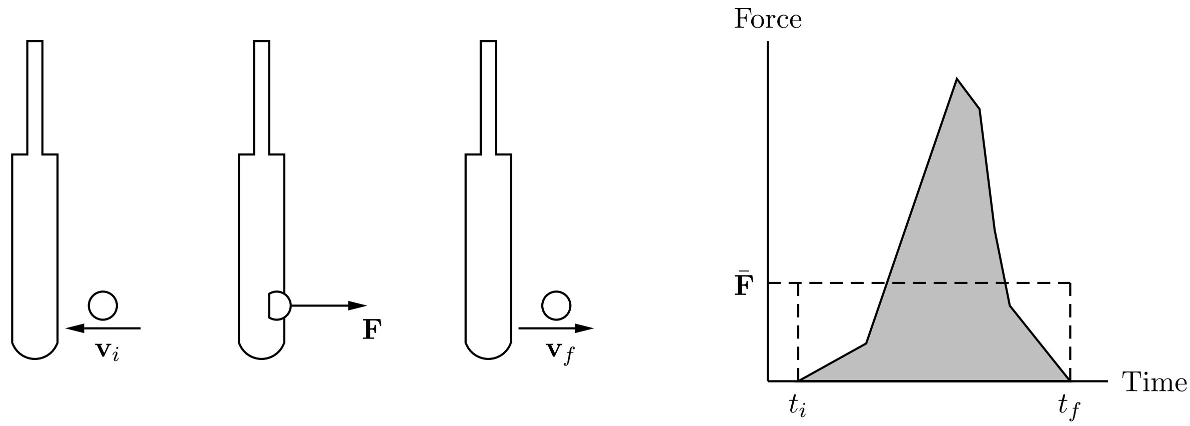
The power can now be found from Equation (50). Thus

$$P = F\bar{v} = 2000\,\mathrm{N} \times 10\,\mathrm{m\,s^{-1}} = 2 \times 10^4\,\mathrm{W}.$$

6 Impulse and momentum

6.1 Impulse

The figure below shows a cricket ball being hit by a bat. The ball's initial velocity is $\mathbf{v}_i$ just before contact is made, and a final velocity $\mathbf{v}_f$ just after leaving the bat. During the time interval $\Delta t = t_f - t_i$ that the ball and bat are in contact, the force exerted on the ball changes in a complicated manner. The graph also shows the magnitude of the average force $\bar{\mathbf{F}}$ between bat and ball.



If the cricket ball is to be struck well, both $\bar{\mathbf{F}}$ and Δt are important. We define the impulse $\mathbf{J}$ of the force as

$$\boxed{\mathbf{J} = \bar{\mathbf{F}}\Delta t}. \tag{51}$$

Impulse equals (average force) $\times$ (contact time). It is a vector having the direction of the average force. The SI units of impulse are N s or kg m s^{-1}.

6.2 Momentum

The **linear momentum p** of an object is the product of the object's mass and velocity.

$$\boxed{\mathbf{p} = m\mathbf{v}}. \tag{52}$$

The SI units of momentum are the same as the impulse $\mathbf{J}$, viz. N s or kg m s^{-1}. Momentum is a **vector quantity**, whose direction is that of the velocity.

6.3 The impulse–momentum theorem

We now recall Newton's second law.

$$F = ma = \frac{m\Delta v}{\Delta t} = \frac{m(v_f - v_i)}{\Delta t}$$

or

$$F\Delta t = mv_f - mv_i = p_f - p_i .$$

That is

$$\boxed{J = p_f - p_i}\,, \tag{53}$$

i.e. (impulse) = (change in momentum of the body). This is the **impulse-momentum** theorem.

6.4 The law of conservation of momentum

Suppose that the net external force applied to some system is zero. Then $J = 0$ and the impulse–momentum theorem implies $p_f = p_i$. This leads us to state the law of conservation of linear momentum

The law of conservation of momentum

If the net external force acting on a system is zero, the total momentum of the system remains constant.

This law, like the law of conservation of energy, is one of the most powerful principles in physics.

6.5 Collisions

A collision is a process (or event) in which the time interval during which the bodies touch is small relative to the total observation time. We can then draw a clear distinction between *'before'* and *'after'*. The law of conservation of momentum is very useful for analyzing collisions because if the system is isolated (i.e. no net external force is acting), momentum is conserved and we can write

$$\boxed{\text{momentum before event}} = \boxed{\text{momentum after event}}.$$

6.5.1 Classification of collisions

Total energy is *always* conserved, but when mechanical energy is converted into heat it is not always possible to apply the energy conservation law in a useful way.

(a) **Elastic collisions** are those in which **kinetic energy is conserved**. Truly elastic collisions can only occur in practice on an atomic scale; even then they are not *always* elastic.

Consider a head-on elastic collision between two objects of mass m_1 and m_2. The initial and final velocities of the objects are u_1, u_2 and v_1 and v_2 respectively. If the target body is initially at rest, $u_2 = 0$. From **conservation of K.E.** we have:

$$\tfrac{1}{2}m_1u_1^2 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 \,. \tag{54}$$

From **conservation of momentum** (which applies in **all** collisions – elastic or otherwise) we have:

$$m_1 u_1 = m_1 v_1 + m_2 v_2 \,. \tag{55}$$

From Equation (54) and Equation (55) we can find the final velocities of the two objects:

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 \tag{56a}$$

$$v_2 = \frac{2m_1}{m_1 + m_2} u_1 . \tag{56b}$$

You are not required to prove Equations (56). It will be expected however that you can apply them in the following **special cases**:

1. If $m_1 = m_2$, then $v_1 = 0$ and $v_2 = u_1$.
2. If $m_1 \gg m_2$, then $v_1 \simeq u_1$ and $v_2 \simeq 2u_1$.

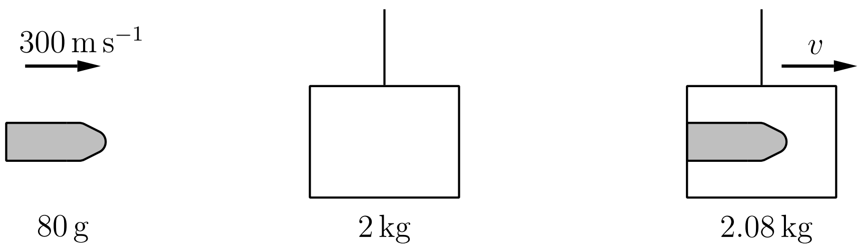
(b) **inelastic collisions** are those in which kinetic energy is not conserved; it may be converted into internal energy (as usually happens), or perhaps elastic potential energy of deformation. On a macroscopic scale this is the most common type of collision.

A **completely inelastic collision** is one in which two bodies stick together after impact (as a bullet being embedded in a target). The loss of kinetic energy is large but not complete.

Note that if the external force is zero, momentum is always conserved **regardless** of whether the collision is elastic or completely inelastic. We conclude that collision-type problems are mostly solved using momentum-conservation techniques.

Example 36: A bullet fired into a block of wood

In a ballistic test, a 2 kg block of wood hangs by a cord of negligible mass and a bullet of mass 80 g is fired with a velocity of 300 m s^{-1} into the block. Calculate the initial velocity with which the block is set in motion.



Solution:

Since there are no external forces acting on the system , we can apply conservation of momentum. After the collision, we consider the bullet and block as a single system with mass $m = 2.08$ kg and velocity v. Hence

$$\begin{aligned} \text{momentum before impact} &= \text{momentum after impact} \\ \therefore \quad m_1 u_1 + m_2 u_2 &= mv \\ \therefore \quad 0.08 \times 300 + 0 &= 2.08 \times v, \end{aligned}$$

which gives

$$v = \frac{0.08 \times 300}{2.08} = +11.5\,\mathrm{m\,s^{-1}}\,.$$

7 Equilibrium of rigid bodies

Up until now, we have been dealing only with forces acting at a common point. Often, the forces acting on a body are **not** applied from a common point but have different "lines of action". In this section we discuss the conditions which need to be satisfied in order for the rigid body to be in equilibrium.

7.1 Centre of gravity

For any rigid body there is a single point through which the resultant of the weights of all particles composing the body acts. This point is called the **centre of gravity** of the body.

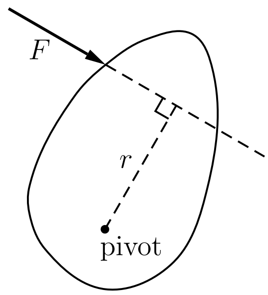
If a body is suspended from any point, the body will come to rest in such a position that its centre of gravity is vertically below the point of suspension. Thus, by suspending a body first from one point and then another, along with a plumb line, the centre of gravity can be located.

Alternatively, the centre of gravity can often be found by balancing.

For regular shaped bodies, the position of the centre of gravity is often obvious from symmetry. For example the centre of gravity of a uniform rod, disc or sphere is located at the mid-point, for triangular lamina at the intersection of the medians and for rectangular lamina at the centre of the diagonals.

7.2 The moment of a force, or the torque about an axis

Besides producing translational motion of an object, when a force acts on a rigid object, it can also cause the object to rotate. The turning effect due to the action of a force is known as **torque**. For simplicity, we will consider only the torque due to coplanar forces acting in a plane perpendicular to the axis of rotation. In general, torque is a vector quantity.



The torque τ (or moment) of a force F about a given axis is given by the product of the force and the **perpendicular** distance of its line of action from the axis.

$$\boxed{\tau = F \times r}\,. \tag{57}$$

The unit of torque is the newton-metre (N m). The distance r is often called the 'lever arm' of the turning effect.

Principle of moments
When a body is in equilibrium under the action of any number of coplanar forces, the algebraic sum of the moments of the forces about **any** point in the plane is zero.

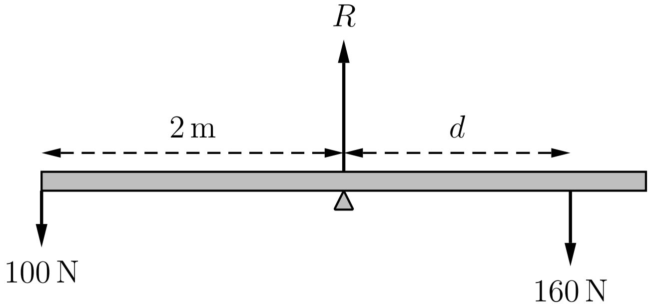
Mathematically:

$$\boxed{\sum \tau = 0}\,. \tag{58}$$

When considering moments about a point, we must choose a positive and negative direction. The convention is that anticlockwise moments are positive and clockwise moments negative. This choice is arbitrary, and the results of any calculation will be the same if the signs are reversed, so long as one is consistent.

Example 37: Moment about an axis

Two children are playing on a see-saw. The total length of the see-saw is 4 m and it is pivoted exactly in the middle. One child weighs 100 N and sits at the end of one side of the see-saw. If the other child weighs 160 N, how far from the other end must she sit so that the see-saw is balanced.



Solution:

Taking moments about the pivot and using Equations (57) and (58):

$$\sum \tau = 100 \times 2 + R \times 0 - 160 \times d = 0,$$

which gives $d = 1.25$ m. The child must therefore sit 75 cm from the other end of the see-saw.

7.2.1 The equilibrium of a rigid body under the action of a system of coplanar forces

Figure 11 shows a rigid bar AB resting on two supports (a trestle table for example). R_1 and R_2 are the reaction (normal) forces at the supports, and W is the weight of the bar (if the bar is uniform the weight acts at the centre of the bar). If the system is in equilibrium (there is no acceleration), then the sum of all the forces, as well as the sum of all the moments must be zero. We can therefore use both Equations (28) and Equation (58).

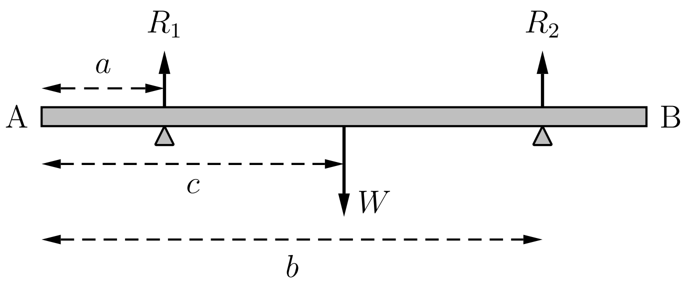


Figure 11: A rigid object in equilibrium.

Taking moments about A,

$$\underbrace{R_1 \times a \quad + \quad R_2 \times b}_{\text{anticlockwise moments}} - \underbrace{W \times c}_{\text{clockwise moments}} = 0.$$

Since the bar is in equilibrium, we also have

$$\sum F_i = R_1 + R_2 - W = 0.$$

For a rigid body to remain in equilibrium when acted on by a set of coplanar forces **two** conditions must be satisfied:

1. The vector sum of all the external forces acting on the body must be zero:

   $$\sum \mathbf{F} = 0.$$

   Because force is a vector quantity, the sum of the x components must be zero, and separately, the sum of the y components must also be zero:

   $$\sum F_x = 0 \text{ and } \sum F_y = 0.$$

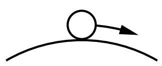
2. The algebraic sum of the moments of all the forces about **any** axis perpendicular to the plane of forces must be zero (Principle of moments):

   $$\sum \tau = 0.$$

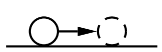
7.2.2 Stable, unstable and neutral equilibrium

These three basic types of equilibrium can be distinguished by giving the body a small displacement. Below are examples for each type of equilibrium for a ball on a surface.

Stable equilibrium: the ball returns to its original position.



Unstable equilibrium: the ball takes up a new position beyond an original small displacement.



Neutral equilibrium: The ball takes up a new position at the end of the displacement.

Note the tendency (where possible) for the centre of gravity to descend to the lowest position, this being the most stable arrangement.

8 Rotational motion

8.1 Angular velocity

If a body (or particle) is rotating with uniform speed v in a circular path we recognize that the particle sweeps through equal angles in equal time intervals. We then define an angular speed ω (omega) given by

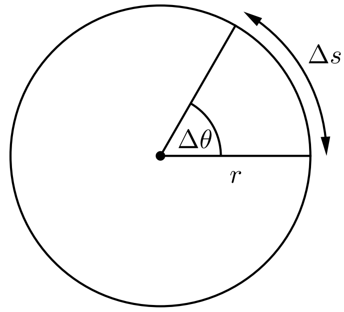
$$\boxed{\omega = \frac{\Delta\theta}{\Delta t}}, \tag{59}$$

where $\Delta\theta$ is the angle (measured in radians) swept out during time Δt. An angle, expressed in radians, is related to an angle in degrees by

$$2\pi \, \text{radians} = 360°.$$

The units of ω are radians per second (rad s^{-1}). (Sometimes ω is expressed in revolutions per minute, rev/s etc.)

An important question is how ω is related to the linear speed v. To answer this we use the definition of an angle. The angle $\Delta\theta$ is defined as (the arc length Δs) $\div$ (circle's radius). i.e.



$$\Delta\theta = \frac{\Delta s}{r} \quad \text{or} \quad r\Delta\theta = \Delta s.$$

Dividing through by Δt gives

$$\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t}.$$

But $\Delta s/\Delta t = v$ and $\Delta\theta/\Delta t = \omega$, so

$$\boxed{v = r\omega}. \tag{60}$$

The concept of **angular velocity** includes both the rate of rotation and the direction of the axis of rotation. Angular velocity is a vector quantity represented by a vector parallel to the axis of rotation. To find the direction of ω, curl the fingers of your right hand in the direction in which the body is rotating. Your thumb then points in the direction of ω.

8.2 Angular acceleration

Angular acceleration α is the rate of change of angular velocity. Just as

$$a = \frac{v-u}{t}, \quad \text{or} \quad v = u + at$$

(assuming a is constant), so

$$\alpha = \frac{\omega - \omega_0}{t}, \quad \text{or} \quad \omega = \omega_0 + \alpha t$$

(assuming α is constant), where ω_0 and ω are the initial and final angular velocities of a rotating body having uniform angular acceleration α.

Since $v = r\omega$,

$$\frac{dv}{dt} = r\frac{d\omega}{dt}$$

or

$$\boxed{a = r\alpha}, \tag{61}$$

for constant r.

8.3 Constant angular acceleration equations of motion

Make the following replacements:

$s \longrightarrow \theta$

$u \longrightarrow \omega_0$

$v \longrightarrow \omega$

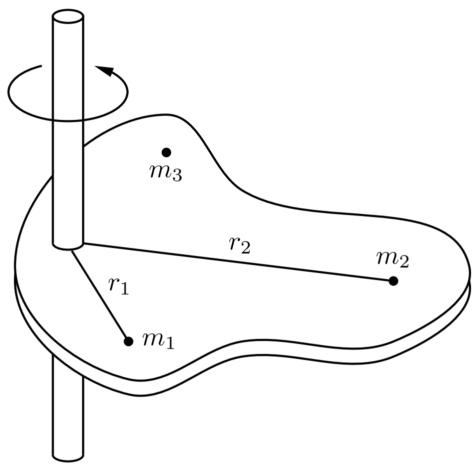
$a \longrightarrow \alpha$ (a and θ constant)

Then

linear motion	angular motion
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = \dfrac{u+v}{2}t$	$\theta = \dfrac{\omega_0 + \omega}{2}t$
$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$

8.4 Newton's second law for rotational motion about a fixed axis

The diagram alongside shows a rigid body rotating about an axis perpendicular to it. Suppose the body is composed of a very large number N of mass particles m_1, m_2, ..., m_N; only three of which are shown for clarity. The torque τ_1 acting on particle m_1 is



$$\begin{aligned}\tau_1 &= F_1 \times (\perp^{\text{r}} \text{ distance from } m_1 \text{ to the axis})\\ &= (m_1 a_1) \times r_1\\ &= m_1 r_1^2 \alpha,\end{aligned}$$

since $a_1 = r_1\alpha$. Similarly $\tau_2 = m_2 r_2^2 \alpha$, $\tau_3 = m_3 r_3^2 \alpha$, etc.

The net torque τ acting on the body is

$$\begin{aligned}\tau &= \tau_1 + \tau_2 + \cdots + \tau_N\\ &= \left(m_1 r_1^2 + m_2 r_2^2 + \cdots + m_N r_N 1^2\right)\alpha,\end{aligned} \tag{62}$$

since α is the same for all particles.

We now define an important quantity: The **moment of inertia** I of a body about an axis is equal to the sum of the products of the mass of each particle in the body and the square of its distance from the axis concerned. Thus

$$\begin{aligned}I &= \left(m_1 r_1^2 + m_2 r_2^2 + \cdots + m_N r_N 1^2\right)\\ &= \sum m_i r_i^2.\end{aligned} \tag{63}$$

The SI unit of I is kg m^2. Substituting Equation (63) into (62) gives:

$$\boxed{\tau = I\alpha}, \tag{64}$$

which is Newton's second law. The derivation above neglects the inter-particle forces between the masses; however the effects of these cancel exactly (because of Newton's third law) and the overall result is the same.

8.5 Rotational kinetic energy and moments of inertia

We refer again to the diagram in Section 8.4. The kinetic energy of mass particle m_1 is

$$\tfrac{1}{2}m_1v_1^2 = \tfrac{1}{2}m_1(r_1\omega)^2 = \tfrac{1}{2}m_1r_1^2\omega^2.$$

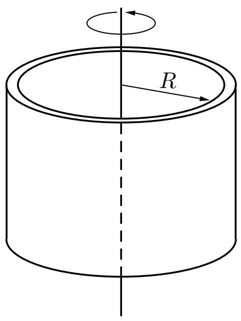
The total kinetic energy of the body is the sum of the kinetic energies of all its particles. Thus

$$E_\mathrm{k} = \tfrac{1}{2}\left(m_1r_1^2 + m_2r_2^2 + \cdots + m_Nr_1^N\right)\omega^2,$$

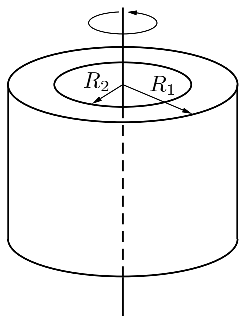
since ω is the same for all particles. The term in brackets is the moment of inertia of the body about the chosen axis. Hence the **rotational kinetic energy** is

$$\boxed{E_\mathrm{k} = \tfrac{1}{2}I\omega^2}. \qquad (65)$$

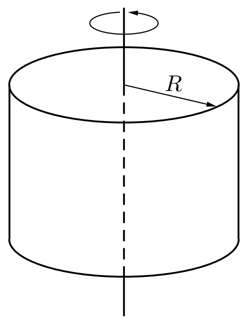
Some important moments of inertia are shown below. You are *not* expected to memorize the formulae.



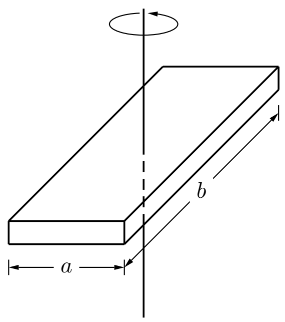
Hoop or cylindrical shell
$I = MR^2$



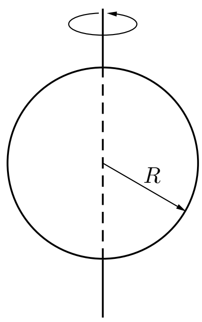
Hollow cylindrical
$I = \frac{1}{2}M\left(R_1^2 + R_2^2\right)$



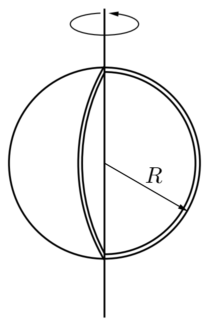
Solid disk or cylindrical
$I = \frac{1}{2}MR^2$



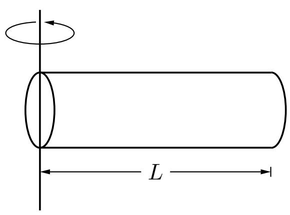
Rectangular plate
$I = \frac{1}{12}M(a^2 + b^2)$



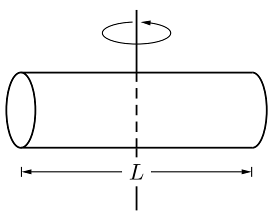
Solid sphere
$I = \frac{2}{5}MR^2$



Thin spherical shell
$I = \frac{2}{3}MR^2$

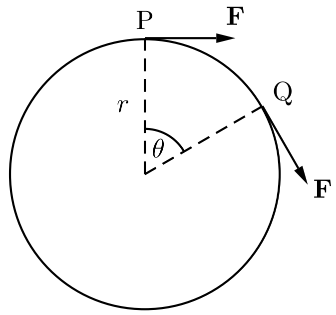


Long thin rod
$I = \frac{1}{3}ML^2$



Long thin rod
$I = \frac{1}{12}ML^2$

8.6 Work and power



Suppose a constant force $\mathbf{F}$ acts tangentially on the rim of a wheel which has a radius r, and that the wheel rotates through an angle θ whilst the force is acting on it. The work done by the force is given by

$$W = Fs,$$

where s is the arclength PQ $= r\theta$. Thus

$$W = Fs = f \times r\theta = (Fr) \times \theta,$$

or

$$\boxed{W = \tau\theta}. \tag{66}$$

Since power $P = \dfrac{dW}{dt}$, for a **constant torque**, we have

$$P = \tau \frac{d\theta}{dt},$$

thus

$$\boxed{P = \tau\omega}. \tag{67}$$

8.7 Angular impulse and momentum

Recall Newton's second law:

$$\tau = I\alpha = I\frac{(\omega - \omega_0)}{t}$$

if the angular acceleration α is constant. So

$$\tau t = I(\omega - \omega_0). \tag{68}$$

We define $I\omega$ to be the angular momentum and τt to be the angular impulse. We will use the symbol L for angular momentum. Hence

$$\boxed{L = I\omega}. \tag{69}$$

The units of angular momentum are $\mathrm{kg\,m^2\,s^{-1}}$.

Notice from Equation (68) that if **no external torque** acts on the system then angular momentum is conserved. We write

$$\boxed{I_1\omega_1 = I_2\omega_2}, \tag{70}$$

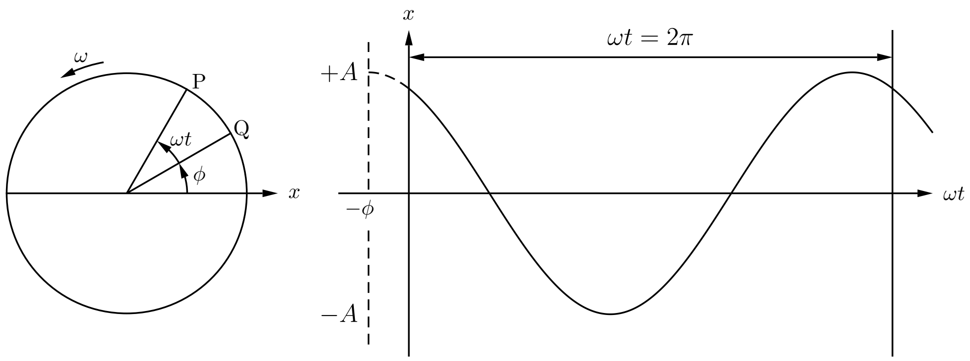
which is the **law of conservation of angular momentum**.

Physical concept	linear quantity/equation	angular quantity/equation
displacement	s	θ
velocity	v	ω
acceleration	a	α
cause of acceleration	force F	torque τ
inertia	mass m	moment of inertia I
Newton's second law	$F = ma$	$\tau = I\alpha$
work	$W = Fs$	$W = \tau\theta$
kinetic energy	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$
momentum	$p = mv$	$L = I\omega$

Table 5: Comparison of linear and angular quantities.

9 Simple harmonic motion

Consider a point P moving with uniform angular speed ω in a circle of radius A. At time $t = 0$ the point is at Q. The angle ϕ is the **initial phase**.



The projection of the position of P on the x axis is given by

$$x = A\cos(\omega t + \phi). \tag{71}$$

Geometric definition of S.H.M.
If a point moves with uniform speed in a circle, its projection on a diameter of the circle moves with S.H.M.

Conversely, if a particle moves in a line in such a way that the coordinate x which specifies its position in the line at any instant t is given by Equation (71), then the particle is moving with S.H.M.

If the point P takes a time T to go round one complete revolution, then the period

$$T = \frac{2\pi}{\omega}. \tag{72}$$

The angular speed ω is also referred to as the angular frequency and is given in terms of the cyclic frequency f by

$$\omega = \frac{2\pi}{T} = 2\pi f. \tag{73}$$

The velocity and acceleration of the point P can be obtained from Equation (71) by differentiating with respect to time. Thus

$$v = \dot{x} = \frac{dx}{dt} = -A\omega \sin(\omega t + \phi) \tag{74}$$

and

$$a = \ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -A\omega^2 \cos(\omega t + \phi). \tag{75}$$

9.1 Relations in S.H.M.

The velocity and acceleration can obtained in terms of the displacement. First square Equation (74), then use the relation $\cos^2\theta + \sin^2\theta = 1$ and Equation (71) to obtain

$$v^2 = \omega^2(A^2 - x^2). \tag{76}$$

From Equations (71) and (75) the acceleration is given by

$$a = -\omega^2 x. \tag{77}$$

Equation (77) shows that the acceleration of a particle moving with S.H.M. is proportional to the displacement and is in the opposite direction. Note that the acceleration in S.H.M. is not constant.

9.2 The force for S.H.M.

Combining Equation (77) with Newton's second law we obtain

$$F = ma = -m\omega^2 x. \tag{78}$$

Equation (78) has the same form as Hooke's law

$$F = -kx, \tag{79}$$

where the spring constant is given by

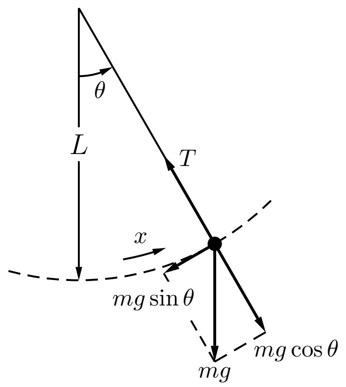
$$k = m\omega^2. \tag{80}$$

By combining Equation (72) with Equation (80) we obtain the period of oscillation of a particle subject to a Hooke's law force:

$$T = 2\pi\sqrt{\frac{m}{k}}. \tag{81}$$

Simple harmonic motion is the motion executed by a particle subject to a force that is proportional to the displacement of the particle but opposite in sign.

9.3 A simple pendulum



An ideal pendulum consists of a point mass m suspended on a massless inelastic string of length L. The displacement $x = L\theta$ is along the arc of the circle of radius L. The restoring force is the component of the weight along the arc:

$$F = -mg\sin\theta. \tag{82}$$

For small oscillations and θ in radians $\theta \approx \sin\theta$. Hence

$$F \approx -mg\theta = -mg\frac{x}{L}. \tag{83}$$

Comparing Equation (83) with Hooke's law (Equation (79)) shows that for small oscillations, an ideal pendulum moves with S.H.M. The period is independent of the mass and is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}. \tag{84}$$

10 Elasticity

10.1 Introduction

The properties of any material are ultimately determined by the type and arrangement of the atoms or molecules which make up the material. In describing the mechanical properties of materials however, a detailed knowledge of the forces between the particles of which a material is composed is not always required. The mechanical properties of materials are those that describe the behaviour of a material when it is exposed to external forces. These properties are of importance when choosing which material to use for building houses, manufacturing cars, the heels of stiletto shoes or a toddler's toy. Important mechanical properties of materials include strength, toughness, stiffness and ductility.

The **strength** of a material describes what forces it can stand before breaking. **Toughness** is a measure of how a material breaks, for example how brittle it is. The **stiffness** describes a material's resistance to deformation and **ductility** relates to how malleable a material is.

In this section we will examine certain mechanical properties of matter which are important in the everyday use of materials and which are readily described from measured quantities.

10.2 Stress and strain

When a force is applied to an object, the object will always deform in some way, even when no change is apparent. In order to make comparisons of the effects of an external force on different materials we introduce the concepts of stress and strain.

Stress
Stress (σ) is the force per unit area applied to a material.

For an applied force F on a cross-sectional area A,

$$\boxed{\text{stress} = \sigma = \frac{F}{A}}. \tag{85}$$

The unit of stress is the $\mathrm{N\,m^{-2}}$ or pascal (Pa).

Strain

Strain (ε) is the fractional deformation of a body.

There are different types of stress which result in different kinds of deformation or strain (see Figure 12). In this course we will only consider longitudinal stress and the corresponding strain. A longitudinal stress which produces an increase in length of a sample is referred to as a **tensile** stress, while a stress that produces a decrease in length is referred to as a **compressive** stress.

For a longitudinal tensile or compressive stress, the corresponding strain is defined as the change in length per unit length. If a stress produces a change of length $\Delta\ell$ in a sample material of original length ℓ, then

$$\boxed{\text{strain} = \varepsilon = \frac{\Delta\ell}{\ell}}. \tag{86}$$

Since strain is a ratio, it has no units. Strain, just like stress, can be either tensile or compressive.

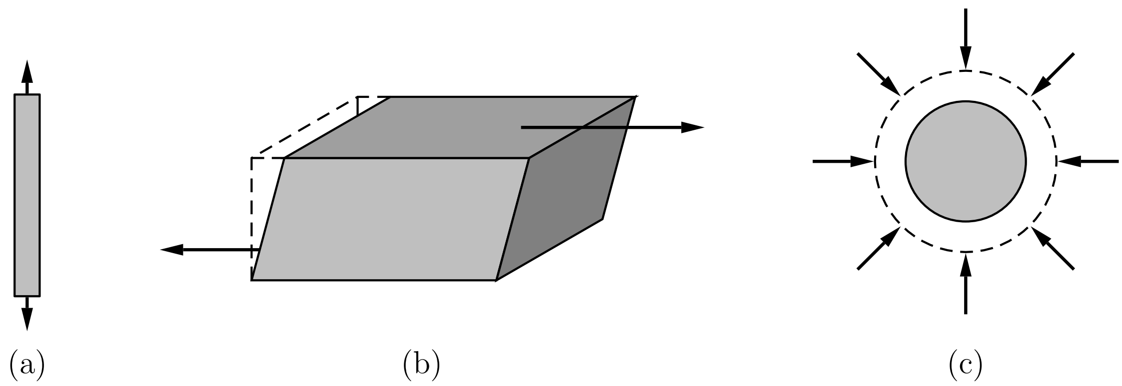


Figure 12: Different kinds of stress. The arrows indicate the applied force F. (a) Longitudinal stress. (b) Shear stress. (c) Bulk stress.

The relation between the stress and the corresponding strain of a material can be determined experimentally. A typical graph of the relation between the tensile stress and strain is depicted in Figure 13.

For a relatively small stress, the relationship between the stress and strain is linear (OA is a straight line in Figure 13). Point A is the linear limit for a material, and up to this point the stress is proportional to the strain (see Section 10.2.2). Between points A and B, the stress is no longer proportional to the strain, however the material will still return to its original length once the stress is removed. The region OB is the **elastic** region and point B is known as the **elastic limit** of the material (see Table 6 below). After point B the deformation is no longer reversible and is now called a **plastic** deformation. A material that has been deformed plastically will not return to its original length when the stress is removed. The dotted line in Figure 13 shows a possible curve for a material that has been stretched beyond the elastic

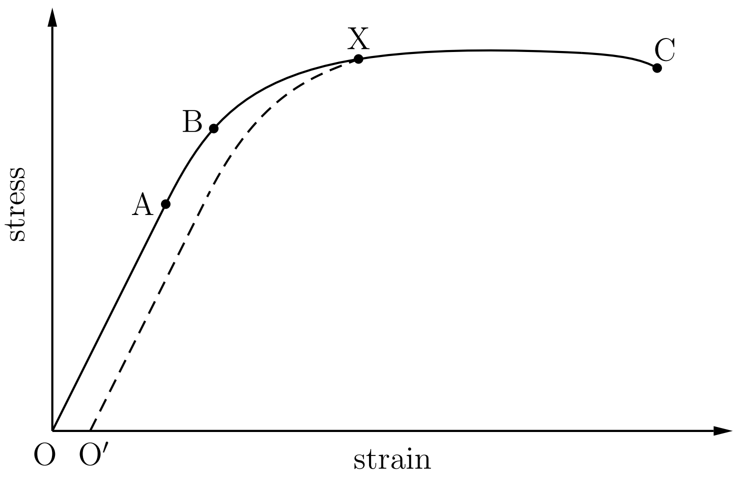


Figure 13: The relation between the tensile stress and strain for a material.

limit. OO′ represents a permanent change of length of the material after the stress has been removed. Point C in Figure 13 represents the point at which the material breaks. BC is known as the plastic region.

For ductile materials like copper, the elastic region is relatively short and the plastic region much longer, whereas for a stiff (brittle) material like glass, the plastic region is very short. A stiff material will break soon after the elastic limit is reached.

For elastic deformations in the linear region (the straight line OA in Figure 13), the constant of proportionality (the slope) is called the **Young's modulus** (Y) for a material and is defined by

$$\boxed{Y = \frac{\text{stress}}{\text{strain}} = \frac{\sigma}{\varepsilon} = \frac{F/A}{\Delta\ell/\ell}\,.} \tag{87}$$

Table 6 lists values of Young's modulus and the elastic limit for some common materials.

	Young's modulus Y $\times 10^{10}$ (Pa)	Elastic limit $\times 10^{8}$ (Pa)
Aluminium	7	1.8
Copper	11	1.5
Steel	20	2.5
Cast iron	19	1.6
Concrete	2	
Bone	1.5	

Table 6: Young's modulus and the elastic limit for various materials

Example 38: Mass suspended from a wire.

A mass of 10 kg is suspended from a wire of length 1.700 m and diameter 2.00 mm. This causes it to stretch by 5.00 mm. Calculate

(a) the stress in the wire,
(b) the strain in the wire, and
(c) Young's modulus for the wire material.

Solution:

(a) The force on the wire causing it to stretch is due to the weight of the 10 kg mass. Hence

$$\sigma = \frac{W}{A} = \frac{mg}{\pi r^2} = \frac{10 \times 9.8}{\pi \times (10^{-3})^2} = 3.12 \times 10^7\,\text{Pa}.$$

(b) From Equation (86):

$$\varepsilon = \frac{\Delta \ell}{\ell} = \frac{5.00 \times 10^{-3}}{1.700} = 2.94 \times 10^{-3}.$$

(c) Young's modulus may be calculated from the definition. Equation (87) gives

$$\text{Y} = \frac{\text{stress}}{\text{strain}} = \frac{3.12 \times 10^7}{2.94 \times 10^{-3}} = 1.06 \times 10^{10}\,\text{Pa}.$$

Example 39: Minimum diameter of a wire under stress.

A steel wire 2.00 metres long supports a load of 15 kg. Calculate the minimum diameter allowable if its extension under this load is not to exceed 3.0 mm. (Young's modulus for steel is 2.0×10^{11} Pa.)

Solution:

A force equal to the weight of the 15 kg mass is the maximum allowable force. Hence the maximum stress allowed is

$$\sigma = \frac{mg}{A_{\text{min}}} = \frac{15\,\text{kg} \times 9.8\,\text{m}\,\text{s}^{-2}}{A_{\text{min}}},$$

where

$$A_{\text{min}} = \pi \left(\frac{d_{\text{min}}}{2}\right)^2.$$

We can determine the stress from Equation (87). Thus

$$\sigma = Y \times \varepsilon = Y \times \frac{\Delta \ell}{\ell} = 2.0 \times 10^{11} \times \frac{3.0 \times 10^{-3}}{2.00} = 3.0 \times 10^8\,\text{Pa}.$$

Hence

$$A_{\text{min}} = \frac{mg}{\sigma} = \frac{15 \times 9.8}{3.0 \times 10^8} = 4.9 \times 10^{-7}\,\text{m}^2,$$

which gives

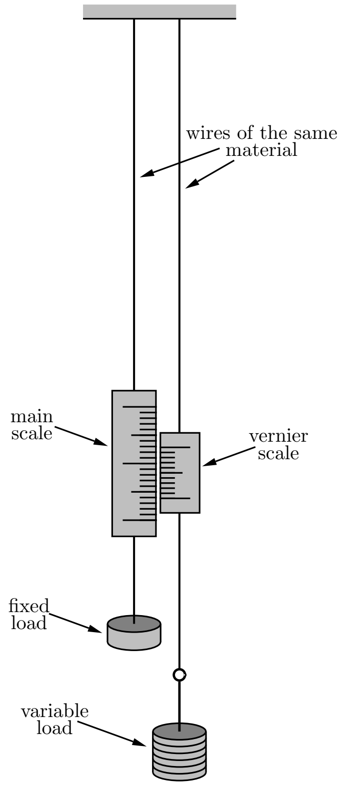
$$d_{\text{min}} = 7.9 \times 10^{-4}\,\text{m} = 0.79\,\text{mm}.$$

10.2.1 Measurement of Young's modulus

Young's modulus for a wire may be measured using apparatus like that depicted in the diagram alongside. Two wires of the same material are suspended from the same support. One wire has a millimetre scale attached to it and the other wire a vernier scale. The wire with the main scale is kept taut by a mass suspended from its end while the other wire is placed under variable tension by adding or removing weights. Wires of the same material are used to eliminate possible errors that could occur due to a change in temperature during the measurements, or possible yield of the support, since these will affect both wires in the same way.

The change in length ($\Delta\ell$) can then be measured for each added (or removed) weight (F) from the vernier scale. The original length (ℓ) is measured with a metre rule and the cross-sectional area (A) of the wire can be determined from the diameter of the wire, measured with a micrometer screw gauge.

Plotting a graph of the extension $\Delta\ell$ versus the load F should give a straight line with slope $\Delta\ell/F$. Young's modulus for the wire can then be obtained by substituting the value of the slope into Equation (87).



10.2.2 Hooke's law

We can rewrite Equation (87) in terms of F and $\Delta\ell$ to obtain

$$F_{\text{applied}} = \left(\frac{YA}{\ell}\right)\Delta\ell.$$

By Newton's third law, the material exerts a force equal and opposite to the force applied on it. This force is called the **restoring** force, as the material tries to restore it's equilibrium configuration. The restoring force is proportional to and in the opposite direction to the extension of the material. Thus

$$\boxed{F_{\text{restoring}} = -k\Delta\ell}, \tag{88}$$

Equation (88) is known as Hooke's law. The constant of proportionality k in Equation (88) is called the **spring constant**. The spring constant is related to Young's modulus by

$$k = \frac{YA}{\ell}. \tag{89}$$

Hooke's law is important in the description of materials that are used for their elastic properties. For example coiled springs and rubber bands obey Hooke's law so long as the extension $\Delta\ell$ is relatively small and stays within the linear portion of the elastic region. As can be seen from Equation (89), the spring constant is larger for a greater cross-sectional area and a smaller length. Short, springs are therefore 'stiffer' than long springs and thick springs are stiffer than thin springs.

Example 40: Compressing and stretching a spring.

A spring with a spring constant $k = 200\,\mathrm{N\,m^{-1}}$ has a length of 8.0 cm when not under load. (a) What force must be applied to compress the spring to half its length? (b) What force must be applied to stretch the spring to twice its length?

Solution:

Hooke's law states that $F = -k\Delta\ell$ where F is the restoring force. The applied force is equal and opposite to the restoring force hence $F_{\text{applied}} = k\Delta\ell$.

(a) When compressed to half its length, the change in length $\Delta\ell = 4.0\,\mathrm{cm}$, hence

$$F_{\text{applied}} = k\Delta\ell = 200 \times 0.040 = 8.0\,\mathrm{N}.$$

(b) When stretched to twice its length, the extension $\Delta\ell = 8.0\,\mathrm{cm}$ and

$$F_{\text{applied}} = k\Delta\ell = 200 \times 0.080 = 16\,\mathrm{N}.$$

11 Fluid dynamics

In Section 4 some properties of fluids at rest were discussed. Here we extend this to a superficial study of the motion of fluids. Fluid dynamics is a vast and difficult subject that is not yet completely understood. It is nevertheless possible to draw some important conclusions and treat a number of useful equations which have a wide applicability.

11.1 Steady versus non-steady flow

Fluid flow can be steady or non-steady. If the fluid velocity at a **given point does not change with time, then the flow is steady**. (This does **not** mean that the velocity is the same at all points in the fluid. Consider two different points A and B in the fluid. The fluid velocity at A will, in general, be different from the fluid velocity at B.) An example of steady flow is water flowing slowly from a garden hose. Examples of non-steady flow are tidal motion, where flow varies periodically with time at a given point, and a waterfall, where the velocity fluctuates erratically.

11.2 Laminar versus turbulent flow

We distinguish two types of fluid flow. If the flow is such that neighbouring layers of the fluid slide by each other smoothly, the flow is said to be **laminar** or **streamline**. In this kind of flow, each particle of the fluid follows a smooth path, called a streamline, see Figure 14(a). In steady flow, streamlines do not cross. Above a certain speed, the flow becomes turbulent. Turbulent flow is characterized by flow in small whirlpool-like circles called eddy currents or eddies. Eddies absorb a great deal of energy and although a certain amount of internal friction, called **viscosity**, is present during streamline flow, it is much greater when the flow is turbulent. (This is one reason why the science of aerodynamics is of crucial importance in the design of modern motor cars, jet liners, etc.)

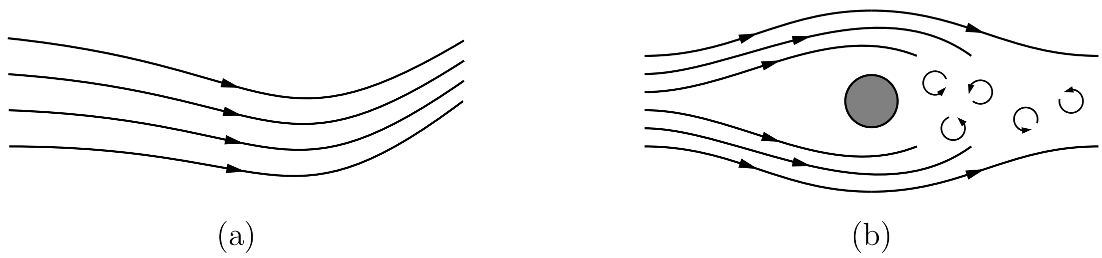
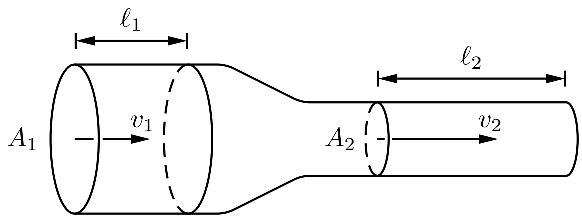


Figure 14: Laminar or streamline flow (a) versus turbulent flow (b).

11.3 Flow rate and an equation of continuity



Let us consider the steady laminar flow of a fluid through an enclosed tube or pipe as shown above. The mass flow rate Q is defined as the mass m of fluid passing a given point per unit time:

$$\boxed{Q = \frac{m}{t}}. \tag{90}$$

The volume V_1 of a fluid passing region (1) in the above pipe in a time t is $V_1 = A_1\ell_1$, where ℓ_1 is the distance the fluid moves in time t. Here A_1 is the cross-sectional area of the tube in region (1). The average velocity of fluid in region (1) is $v_1 = \ell_1/t$, the flow rate Q_1 is thus

$$Q_1 = \frac{m}{t} = \frac{\rho_1 V_1}{t} = \frac{\rho_1 A_1 \ell_1}{t} = \frac{\rho_1 A_1 v_1 t}{t} = \rho_1 A_1 v_1.$$

Likewise, in region (2) $Q_2 = \rho_2 A_2 v_2$. Since no fluid flows in or out of the sides of the pipe, the flow rates in regions (1) and (2) must be equal. Thus

$$Q_1 = Q_2$$

and

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2. \tag{91}$$

Such an equation in physics is called an **equation of continuity**. This one is essentially a statement of mass conservation. The mass of fluid entering a region (1) in a time t must equal the mass of fluid leaving region (2) in the same time period.

If the fluid is incompressible, which is an excellent approximation for liquids under most circumstances (and often for gases as well), then $\rho_1 = \rho_2$ and the equation of continuity becomes

$$A_1 v_1 = A_2 v_2. \tag{92}$$

11.4 Bernoulli's equation

Bernoulli's principle states that: where the velocity of a fluid is high, the pressure is low, and where the velocity is low, the pressure is high.

Bernoulli developed an equation that expresses this principle quantitatively. Bernoulli's equation is

$$P + \tfrac{1}{2}\rho v^2 + \rho g h = \text{constant along a streamline.} \tag{93}$$

Here h represents the height, at some point in the streamline, above some arbitrarily chosen reference position.

This equation is essentially the law of conservation of energy for a moving fluid. Suppose we write the left-hand side of Bernoulli's equation as

$$\frac{1}{V}\left(PV + \tfrac{1}{2}mv^2 + mgh\right),$$

where $V (= m/\rho)$ is the volume of an arbitrary element of fluid, then $\left(\tfrac{1}{2}mv^2 + mgh\right)$ expresses the total mechanical energy of the element and the PV term is due to the work done in moving the element along the streamline.

Strictly, Bernoulli's equation applies only if the following conditions are met:

1. the flow is steady,
2. the flow is laminar,
3. the fluid is non-viscous, and
4. the fluid is incompressible.

A formal derivation of this equation, using the work-energy theorem, can be found in most first-year physics textbooks; although it is not difficult, we will not attempt it here.

11.5 Viscosity

Real fluids have a certain amount of **internal friction** which is called viscosity. It exists in both liquids and gases, and is essentially a frictional force **between different layers** of fluid as these layers move relative to one another. In liquids it is due to the cohesive forces between the molecules; in gases it arises from collisions between the molecules.

Different fluids have different viscosities. Syrup is more viscous than water; grease is more viscous than engine oil; liquids in general are much more viscous than gases. The viscosity of different fluids can be expressed quantitatively by a **coefficient of viscosity** η (eta) which we define below.

11.5.1 Laminar flow in tubes: Poiseuille's law

If a fluid had no viscosity, it could flow through a level pipe without a force being applied. Because of viscosity, a pressure difference between the ends of a tube is necessary for the steady flow of any real fluid, be it water or oil in a pipe, or blood in the circulatory system of a human.

The rate of flow of fluid in a tube depends on the viscosity of the fluid, the pressure difference, and the dimensions of the tube. A Frenchman, J.L. Poiseuille, who was interested in the physics of blood circulation, determined how these variables affect the volume flow rate Q of an incompressible fluid undergoing laminar flow in a cylindrical tube. His result, known as Poiseuille's law, is given by the equation

$$Q = \frac{V}{t} = \frac{\pi r^4 \Delta P}{8\eta L}, \tag{94}$$

where r is the radius of the tube, L is its length, η is the coefficient of viscosity of the fluid, and $\Delta P (= P_2 - P_1)$ is the pressure difference between the ends of the tube.

The derivation of Poiseuille's law requires calculus and we shall not attempt it here. You should remember that the radius enters in Poiseuille's law to the **power of four**.

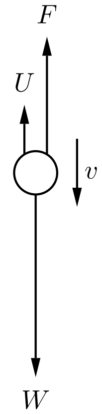
11.5.2 A spherical object moving in a fluid: Stokes' law

One more useful relation in a viscous fluid flow is the expression for the force F exerted on a sphere of radius r moving with speed v in a fluid with viscosity η. When the **flow is laminar**, the relationship is simple. It is

$$F = 6\pi\eta r v; \tag{95}$$

this equation, known as Stokes' law, is stated here without derivation. Strictly, Stokes' law applies only to spherical objects like raindrops falling through air. Note that Stokes' law does **not** apply — regardless of shape — if the flow is turbulent.

If the drag (95) equals the net force accelerating the sphere through the fluid, then the sphere will move with a constant velocity known as the **terminal velocity** or **limiting velocity**.



Suppose the sphere is falling in a gravitational field (e.g. a raindrop falling through air) and the sphere has reached terminal velocity. The net force on the sphere is then

$$W - U - F = 0,$$

where U is the upthrust (buoyancy) on the sphere, W the weight of the sphere, and F the drag due to Stokes' law. Therefore

$$6\pi\eta r v = \tfrac{4}{3}\pi r^3 \rho_{\text{sphere}} g - \tfrac{4}{3}\pi r^3 \rho_{\text{fluid}} g$$

and

$$v = \frac{2r^2 g}{9\eta}\left(\rho_{\text{sphere}} - \rho_{\text{fluid}}\right).$$

11.5.3 Turbulence

In Section 11.2 we discussed the difference between laminar and turbulent flow. Under certain conditions, the character of the flow pattern in a flowing fluid (or a moving object in a stationary fluid) changes. This change could be from laminar to turbulent flow or vice versa. The criterion for deciding whether the flow is laminar or turbulent is the value of a dimensionless quantity called the Reynold's number R_e.

When a fluid flows with velocity v past an object with transverse dimension d, the Reynolds number R_e is defined as

$$R_e = \frac{v\rho d}{\eta}, \tag{96}$$

where ρ is the density of the fluid and η its viscosity.

Reynolds criterion

When $R_e \lesssim 2000$, flow is laminar.

When $R_e \gtrsim 2000$, flow is turbulent.

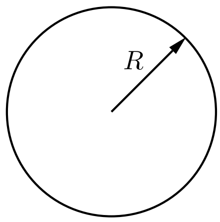
Note that the above are not exact criteria. The transverse dimension d in Equation (96), in typical cases, is taken to be:

When flow is	d
past a sphere	sphere diameter
through a pipe	pipe diameter
across a wing	wing thickness
over a rudder	rudder width
between a sliding plate and a surface	separation of plate from surface

A Revision of some elementary mathematics

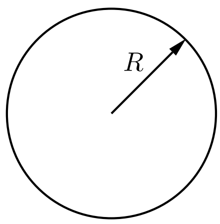
A.1 Geometry

Circle radius R



$$\text{circumference} = 2\pi R$$
$$\text{area} = \pi R^2$$

Sphere radius R

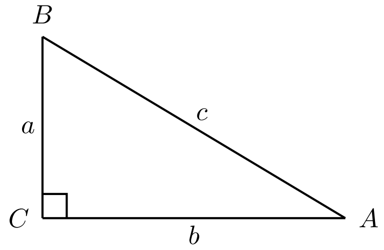


$$\text{area} = 4\pi R^2$$
$$\text{volume} = \tfrac{4}{3}\pi R^3$$

Cylinder length ℓ and radius R

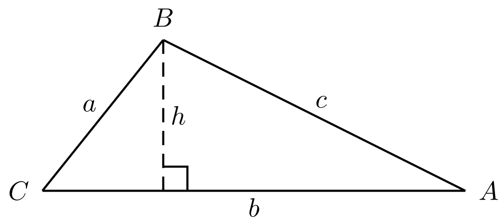
$$\text{surface area} = 2\pi R\ell + 2\pi R^2$$
$$\text{volume} = \pi R^2 \ell$$

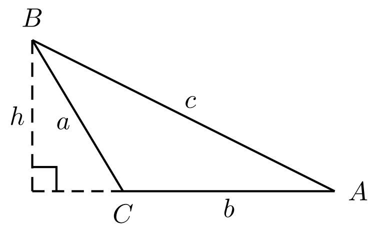
Right-angled triangle ABC



$$\angle A = 90^\circ - \angle B$$
$$c^2 = a^2 + b^2 \qquad \text{(Pythagoras' theorem)}$$
$$\text{area} = \tfrac{1}{2}ab$$

Triangle



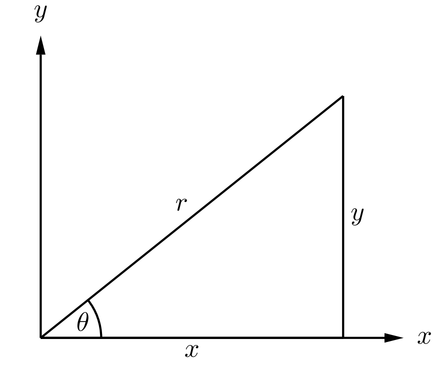


$$\angle A + \angle B + \angle C = 180^\circ \qquad \text{area} = \tfrac{1}{2}\text{base} \times \text{height}$$

A.2 Trigonometry

Definitions

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$



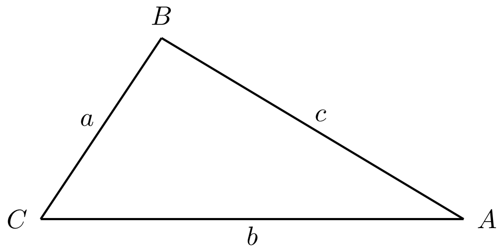
Identities

$$\sin(90^\circ - \theta) = \cos\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta \qquad \sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$
$$\cos(90^\circ - \theta) = \sin\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

Rules for triangles

sine rule: $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

cosine rule: $$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



Quadratic formula

Suppose $ax^2 + bx + c = 0$ where a, b and c are constants independent of the variable x, then the roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exponents

$$\frac{1}{y^n} = y^{-n} \qquad y^n y^m = y^{m+n} \qquad \frac{y^n}{y^m} = y^{n-m}$$
$$y^n z^n = (yz)^n \qquad (y^n)^m = y^{nm}$$

Calculus

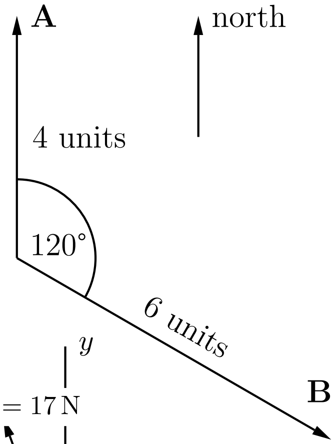
derivatives	anti-derivatives
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1}$
$\frac{d}{dx}(\sin ax) = a\cos ax$	$\int \sin ax\, dx = -\frac{1}{a}\cos ax$
$\frac{d}{dx}(\cos ax) = -a\sin ax$	$\int \cos ax\, dx = \frac{1}{a}\sin ax$

TUTORIAL QUESTIONS

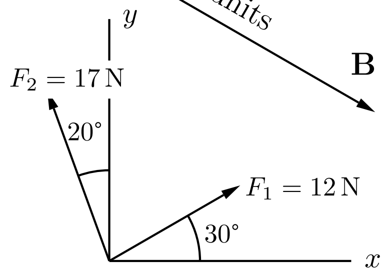
Unless otherwise stated in the question, take the acceleration due to gravity as $g = 9.8\,\mathrm{m\,s^{-2}}$ and the universal gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$.

Basic concepts

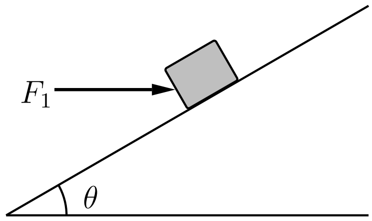
A1 Consider the vectors $\mathbf{A}$ and $\mathbf{B}$ shown in the diagram alongside. Using the head-to-tail method of vector addition, draw a scale diagram to find the magnitude and direction of the vectors
(a) $\mathbf{A} + \mathbf{B}$, and
(b) $\mathbf{A} - \mathbf{B}$.
Use a scale of $1\,\mathrm{cm} \equiv 1\,\mathrm{unit}$.



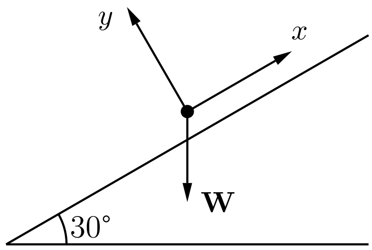
A2 (a) Using trigonometry, resolve the vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ shown opposite into their x and y components.
(b) Hence find (i) $\mathbf{F}_1 + \mathbf{F}_2$, and (ii) $\mathbf{F}_1 - \mathbf{F}_2$.



A3 A horizontal force $\mathbf{F}_1$ is applied to an object on an inclined plane as shown.
(a) Resolve $\mathbf{F}_1$ into components parallel and perpendicular to the plane.
(b) Repeat (a) for a force $\mathbf{F}_2 = -\mathbf{F}_1$.
(c) How do (i) the parallel components of $\mathbf{F}_1$ and $\mathbf{F}_2$ compare, and (ii) the perpendicular components of $\mathbf{F}_1$ and $\mathbf{F}_2$ compare?



A4 A particle with weight $W = 2$ newtons lies on a plane inclined at an angle of 30° to the horizontal. Cartesian axes have been oriented with the x axis parallel to the plane as shown. Resolve W into its x and y components.



A5 An aeroplane pilot sets a compass course due west and maintains an air speed of $240\,\mathrm{km\,h^{-1}}$. After flying for half an hour, he finds himself over a town that is 150 km west and 40 km south of his starting point.
(a) Find the wind velocity, in magnitude and direction.
(b) If the wind velocity were $120\,\mathrm{km\,h^{-1}}$ due south, in what direction should the pilot set his course in order to travel due west? Take the same air speed of $240\,\mathrm{km\,h^{-1}}$.

Equations of motion

B1 Compare your average speed in the following two cases:
(a) You walk 120 m at a speed of $1\,\mathrm{m\,s^{-1}}$ and then run 120 m at a speed of $3\,\mathrm{m\,s^{-1}}$ along a straight track.

(b) You walk for 1 minute at a speed of $1\,\mathrm{m\,s^{-1}}$ and then run for 1 minute at a speed of $3\,\mathrm{m\,s^{-1}}$ along a straight track.

B2 A tennis ball is dropped onto the floor from a height of 4.0 m. It rebounds to a height of 3.0 m. If the ball was in contact with the floor for 0.010 s, calculate its average acceleration during contact.

B3 A train moving between two stations 1100 m apart accelerates uniformly from rest for 40 s, and then moves at constant speed until the brakes are applied resulting in a constant deceleration. If it comes to rest after 30 s and the whole journey takes 90 s, find the maximum speed, the acceleration, and the retardation.

B4 A moving car passes three points A, B and C which are 150 m apart. The time taken to move from A to B was 10 s, and the time taken to move from B to C was 5 s. If the motion of the car was uniformly accelerated, how fast was the car moving as it passed points A, B and C?

B5 A stone is projected vertically upward with a speed of $14\,\mathrm{m\,s^{-1}}$ from a tower 100 m high. Find the maximum height attained and the speed with which it strikes the ground.

B6 A ball rolls off the edge of a tabletop 1 m above the floor, and strikes the floor at a point 1.5 m horizontally from the edge of the table.
(a) Find the time of flight.
(b) Find the initial velocity.
(c) Find the magnitude and direction of the velocity of the ball just before it strikes the floor.

B7 A football is kicked with an initial speed of $22\,\mathrm{m\,s^{-1}}$ at an angle θ above level ground. The ball reaches a maximum height H; and its range is R (maximum horizontal distance).
(a) Calculate the values of H and R for
(i) $\theta = 20°$; (ii) $\theta = 45°$; (iii) $\theta = 70°$.
(b) Sketch these three trajectories roughly to scale.
(c) Use your sketch and a sensible guess to answer the following question: For what value of θ is R a maximum?

B8 A man stands on the roof of a building and throws a ball upwards with a velocity of magnitude $60\,\mathrm{m\,s^{-1}}$ at and angle of 33.0° above the horizontal. The ball leaves his hand at a point 30 m above the ground. Calculate
(a) the maximum height above the roof reached by the ball;
(b) the magnitude of the velocity of the ball just before it strikes the ground;
(c) the horizontal distance from the base of the building to the point where the ball strikes the ground.

B9 An object is projected downward at an angle of 30° to the horizontal, with an initial speed of $40\,\mathrm{m\,s^{-1}}$, from the top of a tower 150 m high. What will be the vertical component of its velocity when it strikes the ground? In what time will it strike the ground? How far from the tower will it strike the ground? At what angle with the horizontal will it strike?

B10 Before leaving the ground, an aircraft moves with constant acceleration and travels 720 m in 12 s from rest. It then leaves the ground. Determine (a) the acceleration,

(b) the speed with which it leaves the ground, (c) the distance covered during the first second and during the twelfth second.

B11 A car driver travelling at $72\,\mathrm{km\,h^{-1}}$ suddenly sees a fallen tree on the road 40 m ahead. He puts on the brakes to stop before he hits the tree. To put on the brakes requires 0.75 s (the reaction time of the driver), after which the retardation is $8\,\mathrm{m\,s^{-2}}$. What is the total stopping time? How far does he travel before the brakes are applied? What is his total stopping distance? If he subsequently travels at twice the speed, how far ahead should he be able to see clearly for safety? (Assume the deceleration is the same.)

B12 A stone is dropped and then 1.0 s later, from a point 5.0 m lower, a second stone is dropped. When will the two stones be 15 m apart?

B13 Four-tenths of a second after bouncing on a trampoline, a gymnast is moving upward with a speed of $6.0\,\mathrm{m\,s^{-1}}$. To what height above the trampoline does the gymnast rise before falling back down?

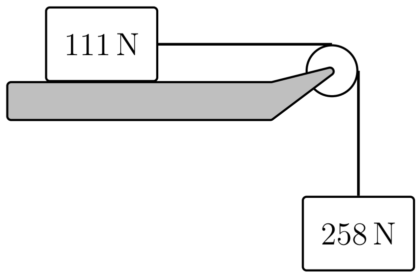
Applications Newton's second law

C1 A 560 N physics student stands on a bathroom scale in an elevator. As the elevator starts moving, the scale reads 800 N.
(a) Find the acceleration of the elevator (magnitude and direction).
(b) What is the acceleration if the scale reads 450 N?
(c) If the scale reads zero, should the student worry? Explain.

C2 A car is towing a trailer. The driver starts from rest and accelerates to a speed of $11\,\mathrm{m\,s^{-1}}$ in a time of 28 s. The mass of the trailer is 410 kg. What is the tension in the hitch that connects the trailer to the car?

C3 A car of mass 1380 kg is moving due east with an initial speed of $27.0\,\mathrm{m\,s^{-1}}$. After 8.00 s the car has slowed down to $17.0\,\mathrm{m\,s^{-1}}$. Find the magnitude and direction of the net force that produces the deceleration.

C4 In the diagram, the weight of the block on the table is 111 N and that of the hanging block is 258 N. Ignoring all frictional effects and assuming the pulley to be massless, find (a) the acceleration of the two blocks and (b) the tension in the cord.

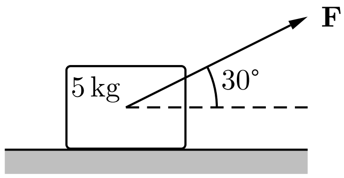


C5 Two objects A and B whose mass is 8.0 kg each are connected by a cord passing over a frictionless pulley. If 0.40 kg is shifted from B to A, (a) what acceleration results and (b) what is the tension in the cord?

C6 A rescue helicopter is lifting a man ($W = 822\,\mathrm{N}$) from a capsized boat by means of a cable and harness. (a) What is the man's mass? (b) What is the tension in the cable when the man is given an initial upward acceleration of $1.10\,\mathrm{m\,s^{-2}}$? (c) What is the tension during the remainder of the rescue when he is pulled upward at a constant velocity?

C7 A lunar landing craft (mass $m = 11\,400\,\text{kg}$) is about to touch down on the surface of the moon, where the acceleration due to gravity is $1.6\,\text{m}\,\text{s}^{-2}$. At an altitude of 165 m the craft's downward velocity is $18.0\,\text{m}\,\text{s}^{-1}$. To slow down the craft, a retrorocket is fired to provide an upward thrust. Assuming the descent is vertical, find the magnitude of the thrust needed to reduce the velocity to zero at the instant when the craft touches the lunar surface.

C8 The 5 kg block shown in the diagram opposite is being pulled to the right at a constant speed by a force **F** which makes an angle of 30° to the horizontal. Given $\mu_k = 0.6$, calculate the magnitude of **F**.



C9 Three identical blocks A, B and C each have mass m. Blocks A and B rest on a horizontal surface. The coefficient of kinetic friction between the blocks and the surface is μ_k. A is attached to B by means of a cord, and B is attached to C by means of a cord passing over a frictionless pulley. Show that (a) the acceleration of the system is $a = \frac{1}{3}g(1-2\mu_k)$, (b) the tensions are $T_1 = \frac{1}{3}mg(1+\mu_k)$ and $T_2 = \frac{2}{3}mg(1+\mu_k)$.

C10 An object weighing 500 N slides down a hill at constant velocity, the angle with the horizontal being 30°. Find (a) the downhill component of the weight, (b) the friction force opposing the motion, (c) the component of the weight normal to the surface, (d) the coefficient of kinetic friction.

C11 A playground slide of constant slope is 4.5 m in length. The upper and lower ends are 3.0 m and 0.5 m vertically above the ground, respectively.
- (a) If a boy starts sliding from rest at the upper end of the slide, find his velocity at the lower end assuming the coefficient of sliding friction to be constant at 0.25.
- (b) What percentage is the final velocity of that which would have been obtained if the friction were negligible?
- (c) For what angle of slope would the boy slide down without acceleration (given a starting push)?

C12 A block rests on an inclined plane that makes an angle θ with the horizontal. The coefficient of sliding friction is 0.50, and the coefficient of static friction is 0.75.
- (a) As the angle θ is increased, find the smallest angle at which the block starts to slip.
- (b) At this angle, find the acceleration once the block has begun to move.
- (c) How much time is required for the block to slip 20 m along the inclined plane?

C13 A 20 kg box is pushed up a rough sloping ramp, inclined at 30° to the horizontal, having a coefficient of kinetic friction of 0.3, by a horizontal force of magnitude 300 N.
- (a) What is the normal force?
- (b) What is the frictional force?
- (c) What is the acceleration of the block?
- (d) If the force is reduced until the acceleration becomes zero, what is the magnitude of the force?

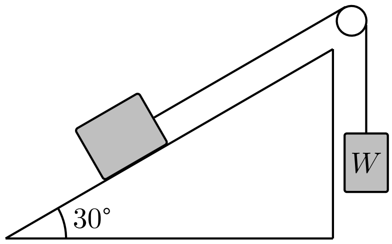
C14 A block weighing 100 N is placed on an inclined plane of slope angle 30° and is connected to a second hanging block of weight W by a cord passing over a small frictionless pulley, as in the figure below. The coefficient of static friction is 0.40 and the coefficient of sliding friction is 0.30.

(a) Find the weight W for which the 100 N block moves up the plane at constant speed.

(b) Find the weight W for which it moves down the plane at constant speed.

(c) For what range of values of W will the block remain at rest?

(d) Does the answer to (c) contradict the answers for (a) and (b)?



C15 A motorcycle goes over the top of a hill. The road may be considered to be an arc of a circle in a vertical plane of radius 88.2 m. With what maximum speed may the motorcycle travel without leaving the road tangentially?

C16 In Bohr's model of the hydrogen atom, an electron (mass 9.11×10^{-31} kg) revolves around a proton in a circular orbit of radius 5.28×10^{-11} m with a speed of $2.18 \times 10^{6}\,\mathrm{m\,s^{-1}}$. Calculate

(a) the period of the electron (i.e. the time taken to complete one revolution).

(b) the acceleration of the electron, and

(c) the centripital force on the electron. What supplies this force?

C17 A car travels without skidding at a speed of $26\,\mathrm{m\,s^{-1}}$ around a curve of radius 92 m on a horizontal road. Calculate the smallest possible value of μ_s between the tyres and the road.

C18 A 0.25 kg mass moves in a vertical circle at the end of a string of length 30 cm. Calculate the tension in the string at the following points: (a) at the top of the circle where the speed is $3.00\,\mathrm{m\,s^{-1}}$, (b) at the bottom of the circle where the speed is $4.56\,\mathrm{m\,s^{-1}}$, and (c) halfway up, where the speed is $3.86\,\mathrm{m\,s^{-1}}$.

C19 At what angle should a curve of radius 150 m be banked so cars travel safely at $25\,\mathrm{m\,s^{-1}}$ without relying on friction?

Gravitation

D1 The mass of the moon is about one eighty-first, and its radius on-fourth, that of the earth. Calculate the acceleration due to gravity on the surface of the moon.

D2 A hypothetical planet has a radius of 500 km and an acceleration due to gravity of $3.0\,\mathrm{m\,s^{-1}}$ at its surface. What is the gravitational acceleration 100 km above its surface? Calculate the mass of the planet.

D3 A satellite orbit has radius 6500 km. If the earth's mass is 5.98×10^{24} kg, calculate the orbital speed of the satellite. Calculate the period of the satellite. Is this satellite in a geosynchronous orbit?

D4 A uniform sphere with mass 0.200 kg is 6.0 m to the left of a second uniform sphere with mass 0.300 kg. Where, in addition to infinitely far away, is the resultant gravitational field due to these masses equal to zero?

D5 At what height **above** the earth's surface will the acceleration due to gravity be $4.90\,\mathrm{m\,s^{-2}}$? The radius of the earth is 6370 km.

Hydrostatics

E1 Numerous jewellery items of silver are melted down and cast into a solid circular disk that is 0.0200 m thick. The total mass of the jewellery is 10.0 kg. Find the radius of the disk. (The density of silver is $1.05 \times 10^4\,\mathrm{kg\,m^{-3}}$.)

E2 An irregularly shaped chunk of concrete has a hollow spherical cavity inside. The mass of the chunk is 33 kg, and the volume enclosed by the outside surface of the chunk is $0.025\,\mathrm{m^3}$. What is the radius of the spherical cavity? (The density of concrete is $2.2 \times 10^3\,\mathrm{kg\,m^{-3}}$)

E3 At times during a walking motion, nearly the entire weight of the body acts on one heel. (a) Calculate the pressure exerted by a woman of mass 55 kg if the heel is circular, with radius of 6.0 mm. (b) How does this compare with the pressure under an elephant's foot? Assume a fully grown elephant of weight 37 000 N standing evenly on all four feet. Treat the feet as circles of diameter 40 cm.

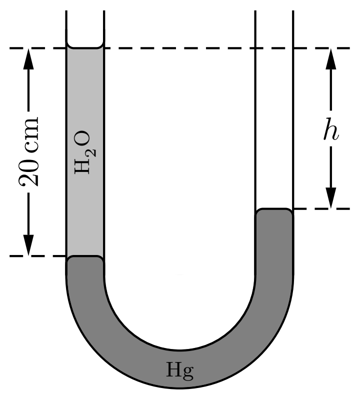
E4 If a barometer reads 76 cmHg on the beach at Durban, what will it read in Pietermaritzburg which is 600 m above sea level, taking the mean density of air as $1.20\,\mathrm{kg\,m^{-3}}$ and the density of mercury as $13\,600\,\mathrm{kg\,m^{-3}}$?

E5 How high can water rise in a pipe if a pressure gauge at the bottom of the pipe shows the excess pressure is 3×10^5 Pa? (Take the density of water as $1000\,\mathrm{kg\,m^{-3}}$.)

E6 The deep end of a swimming pool has a depth of 2.00 m. The atmospheric pressure above the pool is 1.01×10^5 Pa. What is the pressure at the bottom of the pool?

E7 A 1.00 m tall container is filled to the brim, part way with mercury and the rest of the way with water. The container is open to the atmosphere. What must be the depth of each layer, so the absolute pressure on the bottom of the container is twice the atmospheric pressure P_0? (take $P_0 = 76$ cmHg.)

E8 A U-tube is partly filled with equal volumes of water and mercury (which do not mix). If each liquid fills a 20 cm long section of the tube, what is the difference in levels, h, of the two upper surfaces?

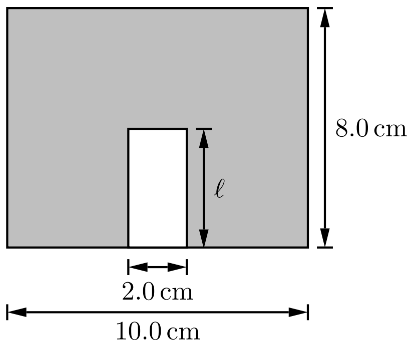


E9 A glass tube in the shape of an upside-down "U" has its ends dipping into beakers containing oil and water respectively. When some air is sucked out of the tube the liquids rise to different levels. The height of the surface of oil in the tube above that in the beaker is 35.5 cm. The corresponding reading for water is 28.4 cm. Find the relative density of the oil. (Explain your working carefully.)

E10 A copper ball (relative density 8.9) has a mass of 267 g. Calculate (a) its density (b) its weight and (c) the upthrust on it when suspended in water.

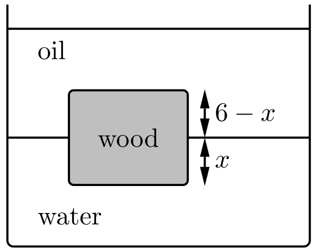
E11 If the RD of ice is 0.92 and that of sea water is 1.03, calculate the total volume of a mass of ice that floats with 1000 m^3 above the water.

E12 A uniform wooden cylinder has a density of 800 $kg\,m^{-3}$, a height of 8.0 cm and a diameter of 10.0 cm. A cylindrical hole of diameter 2.0 cm is drilled part-way up into the cylinder through its base (see diagram). The hole is then filled with lead (density $1.1 \times 10^4\,kg\,m^{-3}$). Calculate the necessary length ℓ of the lead-filled hole if the cylinder is just to submerge when placed in water.



E13 A thin-walled, hard, plastic ball (like a ping-pong ball) has a diameter of 3.8 cm and an average density of 8 $kg\,m^{-3}$. Calculate the force required to hold it completely submerged under water.

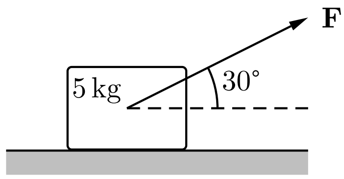
E14 A cylindrical block of wood 6 cm in height has a density of 850 $kg\,m^{-3}$. It is floating in water. Oil ($\rho_{oil} = 800\,kg\,m^{-3}$) is now poured on top of the water, completely submergin the block as shown. Calculate the depth x to which the block is immersed in water.



E15 One kilogram of glass ($\rho = 2600\,kg\,m^{-3}$) is shaped into a hollow spherical shell that just floats in water. Calculate the inner and outer radii of the shell.

Work, energy, power, impulse and momentum

F1 The diagram alongside shows a 5 kg block at rest on a frictionless surface. A force **F** of magnitude 60 N acting at an angle of 30° to the horizontal displaces the block 3 m to the right. Calculate the work done by **F**. Hence (use the work–energy theorem) find the final velocity of the block.



F2 A brick slides on level ground with an initial speed of 28 $m\,s^{-1}$. The coefficient of sliding friction between the brick and the ground is 0.25. Use the work–energy theorem to calculate the distance and time the brick will travel before coming to rest.

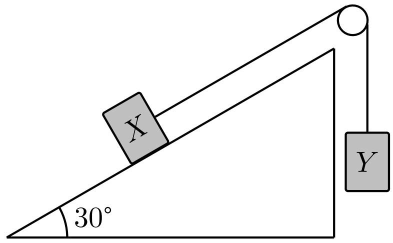
F3 A 1500 kg car is coasting down a 30° hill. At a time when the car's speed is 12 $m\,s^{-1}$ the driver applies the brakes. What force (parallel to the road) must be generated by application of the brakes if the car is to stop after covering 30 m

F4 Calculate the power developed by an 80 kg student who, in 10 seconds runs up stairs that have a vertical height of 5 m.

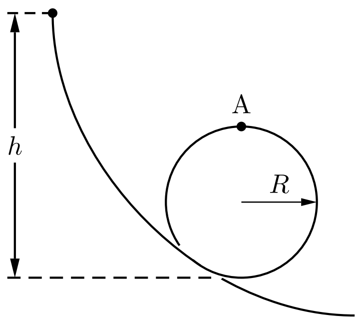
F5 Th output power of an electric motor is 60 kW. At what constant speed can it raise and elevator weighing 1600 N?

F6 A pump is required to lift 800 kg of water per minute from a well 10 m deep and eject it with a speed of $20\,\mathrm{m\,s^{-1}}$. Calculate the power of the pump required.

F7 In the system shown opposite, blocks X and Y have equal mass. Use the **work–energy theorem** to calculate, for the conditions stated in (a) and (b) below, the speed of block X after it has travelled 3 m along the inclined plane. Assume that (a) the system is frictionless and that block X starts from rest, (b) there is friction between block X and the plane. Motion is induced by giving the system a small initial displacement. Take $\mu_k = 0.3$.

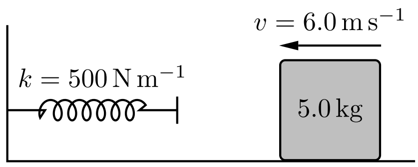


F8 A particle (mass m) slides without friction around a loop-the-loop as shown opposite.
  (a) Suppose $R = 1.5$ m and $h = 7$ m, calculate the speed of the particle at A.
  (b) Calculate the normal force on the particle at A if $m = 5$ g.

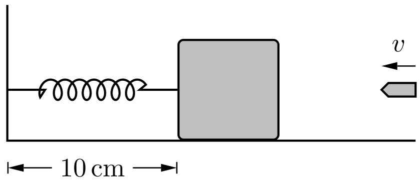


F9 Refer to the figure in Question F8 above. At what height h (expressed in terms of R) will the particle just fail to "loop-the-loop"?

F10 A 5 kg block is moving at $6.00\,\mathrm{m\,s^{-1}}$ along a frictionless horizontal surface towards a spring with force constant $k = 500\,\mathrm{N\,m^{-1}}$ that is attached to a wall. (See the diagram alongside.) Find the maximum distance the spring will be compressed. Assume that the spring has negligible mass.



F11 A rifle bullet of mass 10 g strikes and embeds itself in a block of mass 990 g which rests on a horizontal frictionless surface and is attached to a coil spring, as shown in the figure. The impact compresses the spring 10 cm. Calibration of the spring shows that a force of 1.0 N is required to compress the spring 1 cm.
  (a) Find the maximum potential energy of the spring.
  (b) Find the velocity of the block just after impact.
  (c) What was the initial velocity of the bullet?



F12 A golf ball of mass 0.10 kg initially at rest is given a speed of $50\,\mathrm{m\,s^{-1}}$ when it is struck by a club. If the club and ball are in contact for 2 ms, what average force acted on the ball?

F13 A 150 g cricket ball is hit by a bat. Just before impact, the ball is travelling horizontally towards the right at $40\,\mathrm{m\,s^{-1}}$ and it leaves the bat travelling to the left at an angle of 30° above the horizontal with a speed of $60\,\mathrm{m\,s^{-1}}$. If the bat and ball are in contact for

5×10^{-3} s, find the horizontal and vertical components of the average force exerted by the bat.

F14 A bullet of mass 2 g, travelling in a horizontal direction with a velocity of $500\,\mathrm{m\,s^{-1}}$, is fired into a wooden block of mass 1 kg, initially at rest on a level surface. The bullet passes through the block and emerges with its velocity reduced to $100\,\mathrm{m\,s^{-1}}$. The block slides a distance of 20 cm along the surface from its initila position. Calculate
(a) the speed of the block at the instant after the bullet passed through it;
(b) the coefficient of sliding friction between block and surface.

F15 A 7 g bullet fired into a 2 kg block of soft wood suspended by a long rope, and the bullet remains embedded in the block. The impact causes the centre of gravity of the block to rise 10 cm. Find the initial velocity of the bullet.

F16 A man standing on level ice pushes an object so that it slides away from him. Its mass is 5 kg and the initial speed is $20\,\mathrm{m\,s^{-1}}$. The man has a mass of 80 kg. If the ice has a zero friction coefficient, what happens to the man?

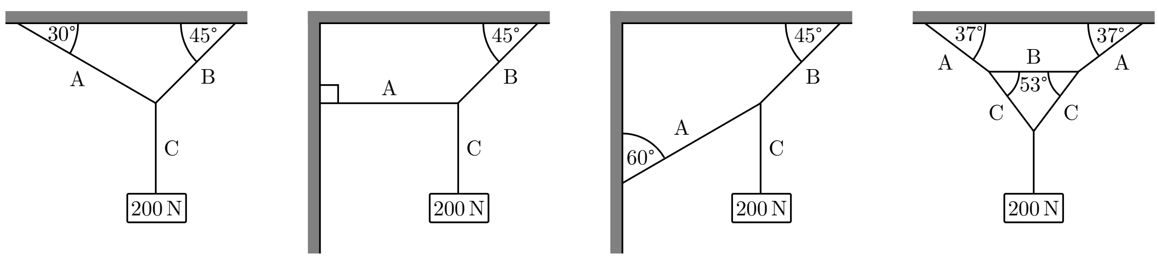
F17 Two blocks of mass 300 g and 200 g are moving toward each other along a horizontal frictionless surface with velocities of $50\,\mathrm{m\,s^{-1}}$ and $100\,\mathrm{m\,s^{-1}}$, respectively.
(a) If the blocks collide and stick together, find their final velocity.
(b) Find the liss of kinetic energy during the collision.
(c) Find the final velocity of each block if the collision is completely elastic.

F18 Two identical balls A and B are rolling towards each other with speeds of $7.0\,\mathrm{m\,s^{-1}}$ and $4.0\,\mathrm{m\,s^{-1}}$, respectively. They undergo a head-on elastic collision. Determine their velocities (magnitude and direction) after the collision.

F19 A particle A of mass 1.0×10^{-27} kg and velocity $5.0 \times 10^{7}\,\mathrm{m\,s^{-1}}$ undergoes a glancing collision with an identical particle B initially at rest. After the collision, the new speed of A is $4.0 \times 10^{7}\,\mathrm{m\,s^{-1}}$. Assuming no loss of kinetic energy in the collision (i.e. that the collision is *elastic*) calculate (a) the angle between the directions of A and B after the collision, (b) the momentum of B after the collision, (c) the directions of A and B after the collision relative to the original direction of A.

Static equilibrium

G1 Find the tension in each of the cords shown in the figure below. The weight of the suspended body is 200 N.



(a) (b) (c) (d)

G2 A uniform rod 1 m long and weighing 30 N is supported in a horizontal position on a fulcrum with weights of 40 N and 50 N suspended from its ends. Calculate the position of the fulcrum.

G3 A uniform metre rule weighing 205 N is supported horizontally by two vertical threads attached to the 40 cm and 70 cm marks respectively. A weight of 100 N is fixed to the rule at the zero end and 300 N at the 100 cm end. Calculate the tension in each thread. If the loaded metre rule is supported by a single thread, where must this be attached in order that the stick shall be horizontal?

G4 A uniform rod of mass 200 kg is hinged at it's lower end. A horizontal cable fixed to its upper end keeps the rod at an angle of 30° to the vertical. Calculate (a) the tension in the cable (b) the horizontal and vertical components of the reaction at the hinge, and hence the resultant magnitude and direction of this reaction.

G5 A uniform ladder 10 m long rests against a vertical frictionless wall with its lower end 6 m from the wall. The ladder weighs 400 N. The coefficient of static friction between the foot of the ladder and the ground is 0.40. A man weighing 800 N climbs slowly up the ladder.

(a) What is the maximum frictional force that the ground can exert on the ladder at its lower end?
(b) What is the actual frictional force when the man has climbed 3 m along the ladder?
(c) How far along the ladder can the man climb before the ladder starts to slip?

G6 An 8.00 m uniform ladder of weight 355 N leans at an angle of 40° to the vertical against a smooth vertical wall. A firefighter of weight 875 N stands 6.30 m from the bottom of the ladder. Calculate the (normal) reaction of the wall on the ladder, and the horizontal and vertical components of the reaction of the ground on the ladder.

Rotation of rigid bodies

H1 A wheel starts from rest and rotates with constant angular acceleration to an angular velocity of $12\,\mathrm{rad\,s^{-1}}$ in a time of 3 s. Calculate (a) the angular acceleration of the wheel and (b) the angle in radians through which it rotates in this time.

H2 A wheel accelerates so that its angular speed increases uniformly from $150\,\mathrm{rad\,s^{-1}}$ to $580\,\mathrm{rad\,s^{-1}}$ in 16 revolutions. Calculate its angular acceleration.

H3 A circular pulley, 4 m in diameter, is mounted so that it can rotate about an axis passing through its centre. One end of a cord which is wound around the pulley is being pulled off with an acceleration of $6\,\mathrm{m\,s^{-2}}$. Calculate

(a) the angular acceleration of the pulley,
(b) the angular speed of the pulley after 10 s, assuming the system started from rest, and
(c) the linear speed of a point on the circumference of the pulley after 10 s.

H4 A wheel undergoes uniform angular acceleration α from rest. It passes through an angular speed of ω_0 and describes a further 800 revolutions in 40 seconds until it reaches an angular speed of 25 revolutions per second.

(a) Calculate (i) ω_0 (ii) α. Determine also (iii) the total number of revolutions described and (iv) the total time taken.

(b) The wheel is now subjected to a braking torque, and experiences an angular retardation of $\dfrac{\pi}{3}\,\mathrm{rad\,s^{-2}}$. Calculate its angular speed after the next 675 revolutions.

H5 Two small spheres, each of mass 12 g, are attached to the ends of a very light, rigid rod 80 cm in length. The system rotates at $25\,\mathrm{rad\,s^{-1}}$ about an axis perpendicular to the rod, and passing through its centre. Calculate

(a) the moment of inertia of the system about its axis, and

(b) the rotational kinetic energy.

H6 A thin steel ring 0.5 m in diameter and mass 6 kg starts from rest at the top of a plane, 4 m long, and inclined at 30° to the horizontal. Suppose the ring rolls down the plane without slipping. At the instant the ring reaches the bottom of the plane, calculate

(a) the total energy,

(b) the linear velocity of the ring's centre of gravity,

(c) the angular velocity,

(d) the linear acceleration of the centre of gravity, and

(e) the angular acceleration.

H7 A 1.0 kg (solid) sphere rolling on a horizontal surface at $20\,\mathrm{m\,s^{-1}}$ comes to the base of an inclined plane which makes an angle of 30° with the horizontal

(a) Calculate the total kinetic energy of the ball when it is at the base of the incline.

(b) How far up the incline will the ball roll?

H8 A wheel of mass $M = 50\,\mathrm{kg}$ and diameter $2R = 80\,\mathrm{cm}$ rotates about its axis at $2000\,\mathrm{rev\,min^{-1}}$. A braking force of 50 N is applied tangentially to the rim of the wheel.

(a) Calculate the moment of inertia of the wheel about its axis $\left(I = \frac{1}{2}MR^2\right)$.

(b) Calculate the applied torque.

(c) Using the concept of angular impulse, calculate the angular velocity of the wheel in revolutions per minute, 40 s after the braking force is applied.

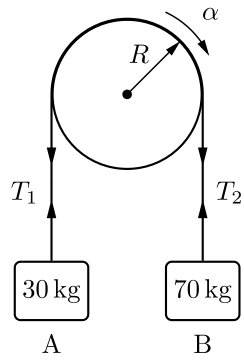
H9 A child pushes with a force of 100 N tangentially to the rim of a playground merry-go-round for 3.00 s. The radius of the merry-go-round is 1.50 m, and its moment of inertia about its axis is $114\,\mathrm{kg\,m^2}$. The initial angular speed of the merry-go-round is $0.500\,\mathrm{rad\,s^{-1}}$.

(a) Calculate (i) the applied torque and (ii) the final angular speed.

(b) The child now gets the merry-go-round up to its final speed of $4.45\,\mathrm{rad\,s^{-1}}$. Then, from a standing position, the 40.0 kg child jumps onto the moving merry-go-round and holds to its rim at a radius of 1.50 m. Calculate the new angular speed.

H10 A string is wound round the horizontal axle, radius 1.50 cm, of a fly wheel and a mass of 200 g is attached to the free end of the string. The system moves from rest until the mass has fallen through 45 cm, when the mass and string are released and the wheel continues to turn at a rate of $\dfrac{10}{\pi}$ revolutions per second. Neglecting friction, calculate (a) the velocity and (b) the acceleration of the mass at the moment of release. Determine also (c) the tension in the string while the mass was descending and (d) the moment of inertia of the flywheel about its axis.

H11 The pulley system in the diagram alongside has a moment of inertia $4.0\,\mathrm{kg\,m^2}$. Block A has a mass 30 kg and block B has a mass 70 kg. Find the angular acceleration α of the pulley and the tensions T_1 and T_2 in the cords when the blocks are released. Assume $R = 0.75\,\mathrm{m}$.



Simple harmonic motion

J1 A vibrating object moves through four complete cycles in 1.00 s. Calculate the frequency f, the angular frequency ω and the period T of the motion.

J2 An object moving with SHM has an acceleration of $0.9\,\mathrm{m\,s^{-2}}$ when it is 0.40 m from its equilibrium position. Calculate the period of the motion.

J3 A body is vibrating with SHM of amplitude 0.20 m and period 0.50 s. Calculate the maximum values of the acceleration and velocity, and the values of acceleration and velocity when the body is 0.10 m away from its force centre. How long does the body take to move from the force centre to a point 0.15 m away?

J4 An object of mass 350 g attached to the end of a spring executes SHM with a period of 1.00 s. The maximum value of the acceleration is $30\,\mathrm{m\,s^{-2}}$. Calculate
(a) the spring constant,
(b) the amplitude of the motion,
(c) the maximum velocity,
(d) the values of the acceleration and velocity when the particle is 0.50 m away from its central position, and
(e) the time taken to move from the central position to a point 0.50 m away.

J5 A body moving with SHM along the x axis has velocities of $20\,\mathrm{m\,s^{-1}}$ and $25\,\mathrm{m\,s^{-1}}$ at distances of 10 m and 8.0 m respectively, from its centre of attraction. Calculate the amplitude of the motion, the period of the motion and the acceleration at a distance of 1.0 m from the centre.

J6 A 0.50 kg object moves with SHM on the end of a horizontal spring with a force constant $k = 300\,\mathrm{N\,m^{-1}}$. When the object is 0.012 m from its equilibrium position, it is observed to have a speed of $0.30\,\mathrm{m\,s^{-1}}$. Neglecting frictional losses, calculate
(a) the total energy of the system,
(b) the amplitude of the motion, and
(c) the maximum speed attained by the object during its motion.

Elasticity

K1 A mass of 170 g is hung on a steel ribbon 750 mm long, 1.9 cm wide and 0.10 mm thick.

(a) Calculate the stress in the ribbon.
(b) If Young's modulus for steel is 2.2×10^{10} Pa, what is the strain in the ribbon?
(c) How much does the ribbon stretch?

K2 A mass of 75 kg is suspended from a steel wire 1.0 mm in diameter.
(a) What is the stress in the wire?
(b) If the wire stretches by 3.0 mm, how long is the wire?

(Young's modulus for steel is 2.0×10^{10} Pa.)

K3 A mass of 10 kg is suspended by a plastic tube of inner diameter 1.8 cm and outer diameter 2.0 cm.
(a) Calculate the stress in the tube.
(b) The tube is 1.0 m long before the mass is attached to it. If Young's modulus is 10^9 Pa, what is the length of the stretched tube?

K4 A straight piece of copper tube 1.0 m long has an internal diameter of 2.0 cm and a wall thickness of 1.0 mm. It is closed at each end. A gas is pumped into the tube until the pressure inside exceeds that outside by 1×10^7 Pa. What is the increase in length of the tube? (Hint: Calculate the force the gas exerts on the plugged ends.) Young's modulus for copper is 1.25×10^{10} Pa.

K5 An aircraft of mass 10^3 kg lands on an aircraft carrier and is decelerated uniformly to rest in 50 m by a steel cable lying along the direction of motion. If the landing speed is 50 m s^{-1} and Young's modulus for steel is 2.0×10^{10} Pa, determine the radius of the smallest cable that will not break (breaking strain of steel is 1.0×10^{-3}).

K6 A child of mass 30 kg is able to swing through an arc of 180° on a swing suspended by two ropes. If the ultimate tensile stress for the rope material is 10^8 Pa, calculate the diameter of the ropes which would provide a safety factor of 5.

Fluids dynamics

L1 Determine whether the flow of water at 3.0 m s^{-1} over a piece of wood of length 30 cm is laminar or not (viscosity of water $\approx 10^{-3}$ Pa s). At what speed does the transition from laminar to turbulent flow take place?

L2 A water drop of radius 0.50 mm falls with a velocity of 5.0 m s^{-1}. Is the air flow past the drop laminar? (Density of air = 1.0 kg m^{-3}, viscosity of air = 1.8×10^{-5} Pa s.) Is the droplet accelerating or not? Can the flow around a falling droplet of this radius ever be turbulent?

L3 Assume that the viscous drag on a sphere moving through a liquid depends on (1) the radius of the sphere, (2) the viscosity of the liquid and (3) the velocity of the sphere. Then use the method of dimensional analysis to show that the drag is proportional to the product $r\eta v$ and hence derive an equation for the limiting velocity of the sphere.

L4 A ball bearing 2.0 mm in diameter is used to determine the viscosity of oil. The ball, falling at its terminal velocity through the oil, is timed between two levels 0.5 m apart at 3.9 s. If the density of steel is 7.8×10^3 kg m^{-3} and that of oil is 8.0×10^2 kg m^{-3}, determine the viscosity of the oil. What is the Reynolds number for the fluid flow?

L5 A bubble of air rises through water. At what bubble radius is the limiting velocity such that the flow of water past the bubble is about to become turbulent? (Viscosity of water = 10^{-3} Pa s, density of air = $1.0\,\mathrm{kg\,m^{-3}}$, density of water = $1000\,\mathrm{kg\,m^{-3}}$.)

L6 A "suspension" is formed by shaking up spherical particles of radius 5.0×10^{-7} m and density $4.0 \times 10^3\,\mathrm{kg\,m^{-3}}$ in water. If the depth of water is 50 mm and the particles are initially distributed uniformly throughout the water, calculate the percentage of particles still in suspension one hour after the mixture has been left to stand. (Viscosity of water = 10^{-3} Pa s.)

L7 An oil drop of density $900\,\mathrm{kg\,m^{-3}}$ reaches a terminal speed of $0.20\,\mathrm{m\,s^{-1}}$ when falling through a gas of viscosity 1.5×10^{-5} Pa s. What is the radius of the drop? What assumptions do you make?

L8 Water flows out of a container through a horizontal pipe of diameter 3.0 mm and length 60 cm. A constant water level 1.0 cm above the pipe is maintained in the container. (Viscosity of water = 10^{-3} Pa s.) How fast is the water in the tube flowing? How long would the stream emerging from the pipe take to fill a 1ℓ container?

L9 Calculate the pressure required to push $6.0 \times 10^2\,\mathrm{mm^3}$ of fluid per second into a tube of internal diameter 0.40 mm and length 3.0 mm. (Viscosity of fluid = 80 Pa s.) Express your answer in atmospheres.

L10 Water flows along a uniform horizontal tube 1.5 m long and radius 1.0 mm. A pressure difference of 5.3 kPa is maintained between the ends. The viscosity of water is 10^{-3} Pa s. Calculate (a) the flow rate in $\mathrm{m^3\,s^{-1}}$, (b) the average water speed.

L11 A horizontal pipe of diameter 3.0 cm has a constriction of diameter 2.0 cm. The flow rate of water in the pipe is $2.0\,\ell\,\mathrm{s^{-1}}$. What is the velocity of the water in the pipe and at the constriction? What is the pressure drop at the constriction? (Density of water = $1000\,\mathrm{kg\,m^{-3}}$.)

L12 Water flows out of a container through a hole of diameter 1.0 cm at a depth of 50 cm below the surface. What volume of water must be supplied per minute to maintain the water level in the container?

L13 Water in a large container has a depth of y_0. There is a hole in the vertical side of the container at a depth $y < y_0$. The jet of water which pours out of the hole hits the level ground on which the container rests at a distance d away from the hole.

(a) Show that $d = 2\sqrt{(y_0 - y)y}$.

(b) For what value of y is d a maximum? (Determine this graphically, if need be.)

(c) What is the y corresponding to $d_{\max}$ for a container 0.50 m deep?

L14 An aircraft wing has a weight of 2.5×10^3 N and an area of $6.0\,\mathrm{m^2}$. The air flows over the top surface at $60\,\mathrm{m\,s^{-1}}$ and over the bottom surface at $50\,\mathrm{m\,s^{-1}}$. The air density = $1.2\,\mathrm{kg\,m^{-3}}$. Calculate the upward force on the wing.

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